THE FOURTH DAY

from

Dialogues Concerning

TWO NEW SCIENCES

GALILEO GALILEI

TRANSLATED BY

Henry Crew & Alfonso de Salvio

WITH AN INTRODUCTION BY

Antonio Favaro

Galileo Galilei, *Dialogues Concerning Two New Sciences*, translated by Henry Crew & Alfonso de Salvio, with an introduction by Antonio Favaro, Dover Publications, Inc., New York,1954:244–295. Originally published in 1904 by the MacMillan company.

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TRANSCRIBER'S NOTES (Added)

The present treatment of Galileo's *Two New Sciences* (1638) follows the condensed format of the 1954 Dover publication with a number of minor cosmetic changes intended to render the work slightly more readable. To this end the *Two New Sciences* has also been split into shorter segments–here in separate parts—the *First* to *Fourth Day* plus the *Introduction* from the Crew & De Salvio translation.

Denoted by {nnn} and [nnn] respectively, both modern and original page numbers have generally been incorporated *within* the text with the former also listed at the top of each page. For additional clarity and general consistency some figures have been redone and/or marginally repositioned.

Lastly, the bi-laterally dissimilar floral spira-form adornments and floral triangles from the title and end pages of the earlier publications have been retained here for necessary completeness.

[*jnh*, 2014]

FOURTH DAY

SALVIATI. Once more, Simplicio is here on time; so let us without delay take up the question of motion. The text of our Author follows:

THE MOTION OF PROJECTILES

In the preceding pages we have discussed the properties of uniform motion and of motion naturally accelerated along planes of all inclinations. I now propose to set forth those properties which belong to a body whose motion is compounded of two other motions, namely, one uniform and one naturally accelerated; these properties, well worth knowing, I propose to demonstrate in a rigid manner. This is the kind of motion seen in a moving projectile; its origin I conceive to be as follows:

> Imagine any particle projected along a horizontal plane without friction; then we know, from what has been more fully explained in the preceding pages, that this particle will move along this same plane with a motion which is uniform and perpetual, provided the plane has no limits. But if the plane is limited and elevated, then the moving particle, which we imagine to be a heavy one, will on passing over the edge of the plane acquire, in addition to its previous uniform and perpetual motion, a downward propensity due to its own weight; so that the resulting motion which I call projection [*projectio*] is compounded of one which is uniform and horizontal and of another which is vertical and naturally accelerated.

We now proceed to {245} demonstrate some of its properties, the first of which is as follows:

[269]

THEOREM I, PROPOSITION I

> A projectile which is carried by a uniform horizontal motion compounded with a naturally accelerated vertical motion describes a path which is a semi-parabola.

SAGR. Here, Salviati, it will be necessary to stop a little while for my sake and, I believe, also for the benefit of Simplicio; for it so happens that I have not gone very far in my study of Apollonius and am merely aware of the fact that he treats of the parabola and other conic sections, without an understanding of which I hardly think one will be able to follow the proof of other propositions depending upon them. Since even in this first beautiful theorem the author finds it necessary to prove that the path of a projectile is a parabola, and since, as I imagine, we shall have to deal with only this kind of curves, it will be absolutely necessary to have a thorough acquaintance, if not with all the properties which Apollonius has demonstrated for these figures, at least with those which are needed for the present treatment.

SAGR. You are quite too modest, pretending ignorance of facts which not long ago you acknowledged as well known—I mean at the time when we were discussing the strength of materials and needed to use a certain theorem of Apollonius which gave you no trouble.

SAGR. I may have chanced to know it or may possibly have assumed it, so long as needed, for that discussion; but now when we have to follow all these demonstrations about such curves we ought not, as they say, to swallow it whole, and thus waste time and energy.

SIMP. Now even though Sagredo is, as I believe, well equipped for all his needs, I do not understand even the elementary terms; for although our philosophers have treated the motion of projectiles, I do not recall their having described the path of a projectile except to state in a general way that it is always a {246} curved line, unless the projection be vertically upwards. But [270] if the little Euclid which I have learned since our previous discussion does not enable me to understand the demonstrations which are to follow, then I shall be obliged to accept the theorems on faith without fully comprehending them.

SALV. On the contrary, I desire that you should understand them from the Author himself, who, when he allowed me to see this work of his, was good enough to prove for me two of the principal properties of the parabola because I did not happen to have at hand the books of Apollonius. These properties, which are the only ones we shall need in the present discussion, he proved in such a way that no prerequisite knowledge was required. These theorems are, indeed, given by Apollonius, but after many preceding ones, to follow which would take a long while. I wish to shorten our task by deriving the first property purely and simply from the mode of generation of the parabola and proving the second immediately from the first.

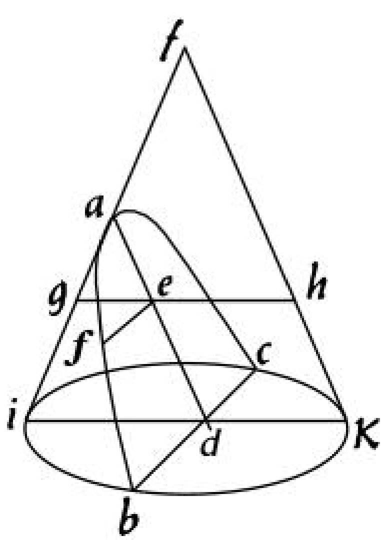

Fig. 106

Beginning now with the first, imagine a right cone, erected upon the circular base *ibkc* with apex at *l*. The section of this cone made by a plane drawn parallel to the side *lk* is the curve which is called a parabola. The base of this parabola *bc* cuts at right angles the diameter *ik* of the circle *ibkc*, and the axis ad is parallel to the side *lk*; now having taken any point *f* in the curve *bfa* draw the straight line *fe* parallel to *bd*; then, I say, the square of *bd* is to the square of *fe* in the same ratio as the axis *ad* is to the portion *ae*. Through the point *e* pass a plane parallel to the circle *ibkc*, producing in the cone a circular section whose diameter is the line *geh*. Since *bd* is at right angles to *ik* in the circle *ibk*, the square of *bd* is equal to the rectangle formed by *id* and *dk*; so also in the upper circle which passes through the points *gfh* the square of *fe* is equal to the rectangle formed by {247} *ge* and *eh*; hence the square of *bd* is to the square of *fe* as the rectangle *id.dk* is to the rectangle *ge.eh*. And since the line *ed* is parallel to *hk*, the line *eh*, being parallel to *dk*, is equal to it; therefore the rectangle *id.dk* is to the rectangle *ge.eh* as *id* is to [271] *ge*, that is, as *da* is to *ae*; whence also the rectangle *id.dk* is to the rectangle *ge.eh*, that is, the square of *bd* is to the square of *fe*, as the axis *da* is to the portion *ae*. Q. E. D.

The other proposition necessary for this discussion we demonstrate as follows. Let us draw a parabola whose axis *ca* is prolonged upwards to a point *d*; from any point *b* draw the line *bc* parallel to the base of the parabola; if now the point *d* is chosen so that *da* = *ca*, then, I say, the straight line drawn through the points *b* and *d* will be tangent to the parabola at *b*. For imagine, if possible, that this line cuts the parabola above or that its prolongation cuts it below, and through any point *g* in it draw the straight line *fge*. And since the square of *fe* is greater than the square of *ge*, the square of *fe* will bear a greater ratio to the square of *bc* than the square of *ge* to that of *bc*; and since, by the preceding proposition, the square of *fe* is to that of *bc* as the line *ea* is to *ca*, it follows that the line *ea* will bear to the line *ca* a greater ratio than the square of ge to that of *bc*, or, than the square of *ed* to that of *cd* (the sides of the triangles *deg* and *dcb* being proportional). But the line *ea* is to *ca*, or *da*, in the same ratio as four times the rectangle *ea.ad* is to four times the square of *ad*, or, what is the same, the square of *cd*, since this is four times the square of *ad*; hence four times the rectangle *ea.ad* bears to the square of *cd* {248} a greater ratio than the square of *ed* to the square of *cd*; but that would make four times the rectangle *ea.ad* greater than the square of *ed*; which is false, the fact being just the opposite, because the two portions *ea* and *ad* of the line *ed* are not equal. Therefore the line *db* touches the parabola without cutting it. Q. E. D.

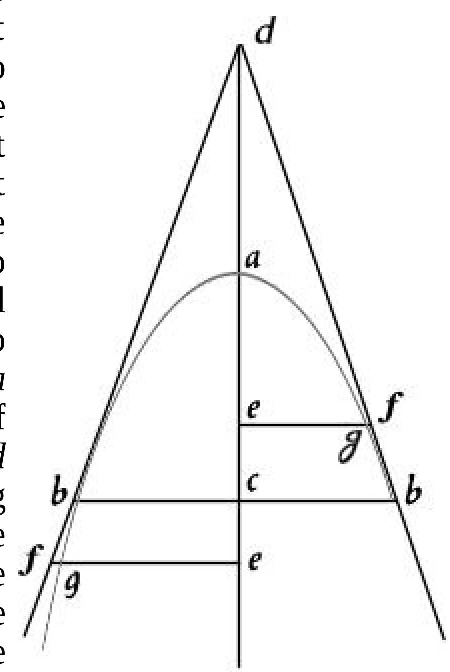

Fig. 107

SIMP. Your demonstration proceeds too rapidly and, it seems to me, you keep on assuming that all of Euclid's theorems are [272] as familiar and available to me as his first axioms, which is far from true. And now this fact which you spring upon us, that four times the rectangle *ea.ad* is less than the square of *de* because the two portions *ea* and *ad* of the line *de* are not equal brings me little composure of mind, but rather leaves me in suspense.

SALV. Indeed, all real mathematicians assume on the part of the reader perfect familiarity with at least the elements of Euclid; and here it is necessary in your case only to recall a proposition of the Second Book in which he proves that when a line is cut into parts is less than that formed on the equal (*i.e.*, less than the square on half the line), by an amount which is the square of the difference between the equal and unequal segments. From this it is clear that the square of the whole line which is equal to four times the square of the half is greater than four times the rectangle of the unequal parts. In order to understand the following portions of this treatise it will be necessary to keep in mind the two elemental theorems from conic sections which we have just demonstrated; and these two theorems are indeed the only ones which the Author uses. We can now resume the text and see how he demonstrates his first proposition in which he shows that a body falling with a motion compounded of a uniform horizontal and a naturally accelerated [*naturale descendente*] one describes a semi-parabola.

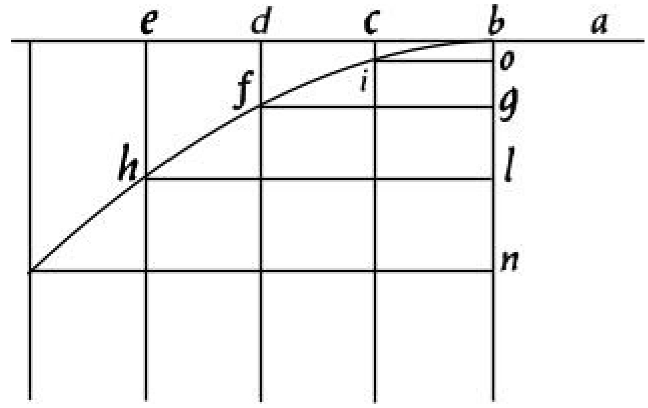

Fig. 108

Let us imagine an elevated horizontal line or plane *ab* along which a body moves with uniform speed from *a* to *b*. Suppose {249} this plane to end abruptly at *b*; then at this point the body will, on account of its weight, acquire also a natural motion downwards along the perpendicular *bn*. Draw the line be along the plane *ba* to represent the flow, or measure, of time; divide this line into a number of segments, *bc, cd, de,* representing equal intervals of time; from the points *b, c, d, e,* let fall lines which are parallel to the perpendicular *bn*. On the first of these lay off any distance *ci*, on the second a distance four times as long, *df*; on [273] the third, one nine times as long, *eh*; and so on, in proportion to the squares of *cb, db, eb,* or, we may say, in the squared ratio of these same lines. Accordingly we see that while the body moves from *b* to *c* with uniform speed, it also falls perpendicularly through the distance *ci*, and at the end of the time-interval *bc* finds itself at the point *l*. In like manner at the end of the time-interval *bd*, which is the double of *bc*, the vertical fall will be four times the first distance *ci*; for it has been shown in a previous discussion that the distance traversed by a freely falling body varies as the square of the time; in like manner the space *eh* traversed during the time be will be nine times *ci*; thus it is evident that the distances *eh, df, cl* will be to one another as the squares of the lines *be, bd, bc.* Now from the points *l, f, h* draw the straight lines *io*, *fg*, *hl* parallel to *be*; these lines *hl, fg, io* are equal to *eb, db* and *cb,* respectively; so also are the lines *bo, bg, bl* respectively equal to *ci, df,* and *eh*. The square of *hl* is to that of *fg* as the line *lb* is to *bg*;

and the square of *fg* is to that of *io* as *gb* is to *bo*; therefore the points *l*, *f*, *h*, lie on one and the same parabola. In like manner it may be shown that, if we take equal time-intervals of any size whatever, and if we imagine the particle to be carried by a similar compound motion, {250} the positions of this particle, at the ends of these time-intervals, will lie on one and the same parabola. Q. E. D.

SALV. This conclusion follows from the converse of the first of the two propositions given above. For, having drawn a parabola through the points *b* and *h*, any other two points, *f* and *l*, not falling on the parabola must lie either within or without; consequently the line *fg* is either longer or shorter than the line which terminates on the parabola. Therefore the square of *hl* will not bear to the square of *fg* the same ratio as the line lb to bg, but a greater or smaller; the fact is, however, that the square of *hl* does bear this same ratio to the square of *fg*. Hence the point *f* does lie on the parabola, and so do all the others.

SAGR. One cannot deny that the argument is new, subtle and conclusive, resting as it does upon this hypothesis, namely, that the horizontal motion remains uniform, that the vertical motion continues to be accelerated downwards in proportion to the square of the time, and that such motions and velocities as these combine without altering, disturbing, or hindering each other,* so that as the motion proceeds the path of the projectile does not change into a different curve: but this, in my opinion, [274] is impossible. For the axis of the parabola along which we imagine the natural motion of a falling body to take place stands perpendicular to a horizontal surface and ends at the center of the earth; and since the parabola deviates more and more from its axis no projectile can ever reach the center of the earth or, if it does, as seems necessary, then the path of the projectile must transform itself into some other curve very different from the parabola.

SIMP. To these difficulties, I may add others. One of these is that we suppose the horizontal plane, which slopes neither up nor down, to be represented by a straight line as if each point on this line were equally distant from the center, which is not the case; for as one starts from the middle [of the line] and goes toward either end, he departs farther and farther from the center [of the earth] and is therefore constantly going uphill. Whence it follows that the motion cannot remain uniform {251} through any distance whatever, but must continually diminish. Besides, I do not see how it is possible to avoid the resistance of the medium which must destroy the uniformity of the horizontal motion and change the law of acceleration of falling bodies. These various difficulties render it highly improbable that a result derived from such unreliable hypotheses should hold true in practice.

SALV. All these difficulties and objections which you urge are so well founded that it is impossible to remove them; and, as for me, I am ready to admit them all, which indeed I think our author would also do. I grant that these conclusions proved in the abstract will be different when applied in the concrete and will be fallacious to this extent, that neither will the horizontal motion be uniform nor the natural acceleration be in the ratio assumed, nor the path of the projectile a parabola, etc. But, on the other hand, I ask you

* A very near approach to Newton's Second Law of Motion. [*Trans.*]

not to begrudge our Author that which other eminent men have assumed even if not strictly true. The authority of Archimedes alone will satisfy everybody. In his Mechanics and in his first quadrature of the parabola he takes for granted that the beam of a balance or steelyard is a straight line, every point of which is equidistant from the common center of all heavy bodies, and that the cords by which heavy bodies are suspended are parallel to each other.

Some consider this assumption permissible because, in practice, our instruments and the distances involved are so small in comparison with the enormous distance from the center of the earth that we may consider a minute of arc on a great circle as a straight line, and may regard the perpendiculars let fall from its two extremities as parallel. For if in actual practice one had to [275] consider such small quantities, it would be necessary first of all to criticise the architects who presume, by use of a plumbline, to erect high towers with parallel sides. I may add that, in all their discussions, Archimedes and the others considered themselves as located at an infinite distance from the center of the earth, in which case their assumptions were not false, and therefore their conclusions were absolutely correct. When we {252} wish to apply our proven conclusions to distances which, though finite, are very large, it is necessary for us to infer, on the basis of demonstrated truth, what correction is to be made for the fact that our distance from the center of the earth is not really infinite, but merely very great in comparison with the small dimensions of our apparatus. The largest of these will be the range of our projectiles—and even here we need consider only the artillery—which, however great, will never exceed four of those miles of which as many thousand separate us from the center of the earth; and since these paths terminate upon the surface of the earth only very slight changes can take place in their parabolic figure which, it is conceded, would be greatly altered if they terminated at the center of the earth.

As to the perturbation arising from the resistance of the medium this is more considerable and does not, on account of its manifold forms, submit to fixed laws and exact description. Thus if we consider only the resistance which the air offers to the motions studied by us, we shall see that it disturbs them all and disturbs them in an infinite variety of ways corresponding to the infinite variety in the form, weight, and velocity of the projectiles. For as to velocity, the greater this is, the greater will be the resistance offered by the air; a resistance which will be greater as the moving bodies become less dense [*men gravi*]. So that although the falling body ought to be displaced [*andare accelerandosi*] in proportion to the square of the duration of its motion, yet no matter how heavy the body, if it falls from a very considerable height, the resistance of the air will be such as to prevent any increase in speed and will render the motion [276] uniform; and in proportion as the moving body is less dense [*men grave*] this uniformity will be so much the more quickly attained and after a shorter fall. Even horizontal motion which, if no impediment were offered, would be uniform and constant is altered by the resistance of the air and finally ceases; and here again the less dense [*piu leggiero*] the body the quicker the process. Of these properties [*accidenti*] of weight, of velocity, and also of form [*figura*], infinite in number, it is not possible to give {253} any exact description;

hence, in order to handle this matter in a scientific way, it is necessary to cut loose from these difficulties; and having discovered and demonstrated the theorems, in the case of no resistance, to use them and apply them with such limitations as experience will teach. And the advantage of this method will not be small; for the material and shape of the projectile may be chosen, as dense and round as possible, so that it will encounter the least resistance in the medium. Nor will the spaces and velocities in general be so great but that we shall be easily able to correct them with precision.

In the case of those projectiles which we use, made of dense [*grave*] material and round in shape, or of lighter material and cylindrical in shape, such as arrows, thrown from a sling or crossbow, the deviation from an exact parabolic path is quite insensible. Indeed, it you will allow me a little greater liberty, I can show you, by two experiments, that the dimensions of our apparatus are so small that these external and incidental resistances, among which that of the medium is the most considerable, are scarcely observable.

I now proceed to the consideration of motions through the air, since it is with these that we are now especially concerned; the resistance of the air exhibits itself in two ways: first by offering greater impedance to less dense than to very dense bodies, and secondly by offering greater resistance to a body in rapid motion than to the same body in slow motion.

Regarding the first of these, consider the case of two balls having the same dimensions, but one weighing ten or twelve times as much as the other; one, say, of lead, the other of oak, both allowed to fall from an elevation of 150 or 200 cubits.

Experiment shows that they will reach the earth with slight difference in speed, showing us that in both cases the retardation caused by the air is small; for if both balls start at the same moment and at the same elevation, and if the leaden one be slightly retarded and the wooden one greatly retarded, then the former ought to reach the earth a considerable distance in advance of the latter, since it is ten times as heavy. But this {254} [277] does not happen; indeed, the gain in distance of one over the other does not amount to the hundredth part of the entire fall. And in the case of a ball of stone weighing only a third or half as much as one of lead, the difference in their times of reaching the earth will be scarcely noticeable. Now since the speed [*impeto*] acquired by a leaden ball in falling from a height of 200 cubits is so great that if the motion remained uniform the ball would, in an interval of time equal to that of the fall, traverse 400 cubits, and since this speed is so considerable in comparison with those which, by use of bows or other machines except fire arms, we are able to give to our projectiles, it follows that we may, without sensible error, regard as absolutely true those propositions which we are about to prove without considering the resistance of the medium.

Passing now to the second case, where we have to show that the resistance of the air for a rapidly moving body is not very much greater than for one moving slowly, ample proof is given by the following experiment. Attach to two threads of equal length—say four or five yards—two equal leaden balls and suspend them from the ceiling; now pull them aside from the perpendicular, the one through 80 or more degrees, the other through not more than four or five degrees; so that, when set free, the one falls, passes through the perpendicular, and describes large but slowly decreasing arcs of 160, 150, 140 degrees, etc.; the other swinging through small and also diminishing arcs of 10, 8, 6, degrees, etc.

In the first place it must be remarked that one pendulum passes through its arcs of 180°, 160°, etc., in the same time that the other swings through its 10°, 8°, etc., from which it follows that the speed of the first ball is 16 and 18 times greater than that of the second. Accordingly, if the air offers more resistance to the high speed than to the low, the frequency of vibration in the large arcs of 180° or 160°, etc., ought to be less than in the small arcs of 10°, 8°, 4°, etc., and even less than in arcs of 2°, or 1°; but this prediction is not verified by experiment; because if two persons start to count the vibrations, the one the large, the other the small, they will discover that after counting tens {255} and even hundreds they will not differ by a single vibration, not even by a fraction of one.[278]

This observation justifies the two following propositions, namely, that vibrations of very large and very small amplitude all occupy the same time and that the resistance of the air does not affect motions of high speed more than those of low speed, contrary to the opinion hitherto generally entertained.

SAGR. On the contrary, since we cannot deny that the air hinders both of these motions, both becoming slower and finally vanishing, we have to admit that the retardation occurs in the same proportion in each case. But how? How, indeed, could the resistance offered to the one body be greater than that offered to the other except by the impartation of more momentum and speed [*impeto e velocità*] to the fast body than to the slow? And if this is so the speed with which a body moves is at once the cause and measure [*cagione e misura*] of the resistance which it meets. Therefore, all motions, fast or slow, are hindered and diminished in the same proportion; a result, it seems to me, of no small importance.

SALV. We are able, therefore, in this second case to say that the errors, neglecting those which are accidental, in the results which we are about to demonstrate are small in the case of our machines where the velocities employed are mostly very great and the distances negligible in comparison with the semidiameter of the earth or one of its great circles.

SIMP. I would like to hear your reason for putting the projectiles of fire arms, *i.e.*, those using powder, in a different class from the projectiles employed in bows, slings, and crossbows, on the ground of their not being equally subject to change and resistance from the air.

SALV. I am led to this view by the excessive and, so to speak, supernatural violence with which such projectiles are launched; for, indeed, it appears to me that without exaggeration one might say that the speed of a ball fired either from a musket or from a piece of ordnance is supernatural. For if such a ball be allowed to fall from some great elevation its speed will, owing to the {256} resistance of the air, not go on increasing indefinitely; that which happens to bodies of small density in falling through short distances—I mean the reduction of their motion to uniformity—will also happen to a ball of iron or lead after it has fallen a few thousand cubits; this terminal or final speed [*terminata velocità*] is the maximum which such a heavy body can naturally acquire [279] in falling through the air. This speed I estimate to be much smaller than that impressed upon the ball by the burning powder.

An appropriate experiment will serve to demonstrate this fact. From a height of one hundred or more cubits fire a gun [*archibuso*] loaded with a lead bullet, vertically downwards upon a stone pavement; with the same gun shoot against a similar stone from a distance of one or two cubits, and observe which of the two balls is the more flattened. Now if the ball which has come from the greater elevation is found to be the less flattened of the two, this will show that the air has hindered and diminished the speed initially imparted to the bullet by the powder, and that the air will not permit a bullet to acquire so great a speed, no matter from what height it falls; for if the speed impressed upon the ball by the fire does not exceed that acquired by it in falling freely [*naturalmente*] then its downward blow ought to be greater rather than less.

This experiment I have not performed, but I am of the opinion that a musket-ball or cannon-shot, falling from a height as great as you please, will not deliver so strong a blow as it would if fired into a wall only a few cubits distant, *i.e.*, at such a short range that the splitting or rending of the air will not be sufficient to rob the shot of that excess of supernatural violence given it by the powder.

The enormous momentum [*impeto*] of these violent shots may cause some deformation of the trajectory, making the beginning of the parabola flatter and less curved than the end; but, so far as our Author is concerned, this is a matter of small consequence in practical operations, the main one of which is the preparation of a table of ranges for shots of high elevation, giving the distance {257} attained by the ball as a function of the angle of elevation; and since shots of this kind are fired from mortars [*mortari*] using small charges and imparting no supernatural momentum [*impeto sopranaturale*] they follow their prescribed paths very exactly.

But now let us proceed with the discussion in which the Author invites us to the study and investigation of the motion of a body [*impeto del mobile*] when that motion is compounded of two others; and first the case in which the two are uniform, the one horizontal, the other vertical.]

[280]

THEOREM II, PROPOSITION II

When the motion of a body is the resultant of two uniform motions, one horizontal, the other perpendicular, the square of the resultant momentum is equal to the sum of the squares of the two component momenta.*

Let us imagine any body urged by two uniform motions and let *ab* represent the vertical displacement, while *bc* represents the displacement which, in the same interval of time, takes place in a horizontal direction. If then the distances *ab* and *bc* are traversed,

* In the original this theorem reads as follows

"*Si aliquod mobile duplici motu aquabili moveatur, nempe orizontali et perpendicularis, impetus seu momentum lationis ex utroque motu composita erit potentia aquakis ambobus momentis priorum motuum.*"

For the justification of this translation of the word "potentia" and of the use of the adjective "resultant" see p. 288 below. [*Trans.*]

during the same time-interval, with uniform motions the corresponding momenta will be to each other as the distances *ab* and *bc* are to each other; but the body which is urged by these two motions describes the diagonal *ac*; its momentum is proportional to *ac*. Also the square of *ac* is equal to the sum of the squares of *ab* and *bc*. Hence the square of the resultant momentum is equal to the sum of the squares of the two momenta *ab* and *bc*.

Q. E. D.

SIMP. At this point there is just one slight difficulty which needs to be cleared up; for it seems to me that the conclusion {258} just reached contradicts a previous proposition* in which it is claimed that the speed [*impeto*] of a body coming from *a* to *b* is equal to that in coming from *a* to *c*; while now you conclude that the speed [*impeto*] at *c* is greater than that at *b*.

SALV. Both propositions, Simplicio, are true, yet there is a great difference between them. Here we are speaking of a body urged by a single motion which is the resultant of two uniform motions, while there we were speaking of two bodies each urged with naturally accelerated motions, one along the vertical ab the other along the inclined plane ac. Besides the time-intervals were there not supposed to be equal, that along the incline ac being greater than that along the vertical ab; but the motions of which we now speak, those along ab, bc, ac, are uniform and simultaneous.

SIMP. Pardon me; I am satisfied; pray go on.

[281]

SALV. Our Author next undertakes to explain what happens when a body is urged by a motion compounded of one which is horizontal and uniform and of another which is vertical but naturally accelerated; from these two components results the path of a projectile, which is a parabola. The problem is to determine the speed [*impeto*] of the projectile at each point. With this purpose in view our Author sets forth as follows the manner, or rather the method, of measuring such speed [*impeto*] along the path which is taken by a heavy body starting from rest and falling with a naturally accelerated motion.

THEOREM III, PROPOSITION III

Let the motion take place along the line *ab*, starting from rest at *a*, and in this line choose any point *c*. Let ac represent the time, or the measure of the time, required for the body to fall through the space *ac*; let *ac* also represent the velocity [*impetus seu momentum*] at *c* acquired by a fall through the distance *ac*. In the line *ab* select any other point *b*. The problem now is to determine the velocity at *b* acquired by a body in falling through the distance *ab* and to express this in terms of the velocity at *c*, the measure of which is the length *ac*. Take as a mean proportional between *ac* and *ab*. {259} We shall prove that the velocity at *b* is to that at *c* as the length as is to the length ac. Draw the horizontal line *cd*, having twice the length of *ac*, and *be*,

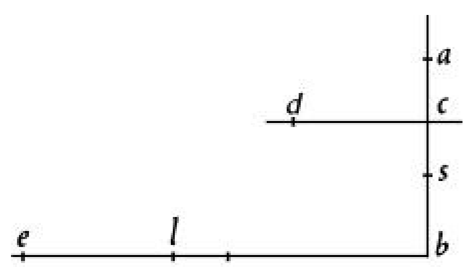

Fig. 110

* See p. 169 above. [*Trans.*]

having twice the length of *ba*. It then follows, from the preceding theorems, that a body falling through the distance *ac*, and turned so as to move along the horizontal *cd* with a uniform speed equal to that acquired on reaching *c* [282] will traverse the distance *cd* in the same interval of time as that required to fall with accelerated motion from *a* to *c*. Likewise be will be traversed in the same time as *ba*. But the time of descent through *ab* is *as*; hence the horizontal distance be is also traversed in the time *as*. Take a point l such that the time as is to the time ac as be is to *bl*; since the motion along *be* is uniform, the distance *bl*, if traversed with the speed [*momentum celeritatis*] acquired at *b*, will occupy the time *ac*; but in this same time-interval, *ac*, the distance *cd* is traversed with the speed acquired in *c*. Now two speeds are to each other as the distances traversed in equal intervals of time. Hence the speed at *c* is to the speed at *b* as *cd* is to *bl*. But since *dc* is to be as their halves, namely, as *ca* is to *ba*, and since *be* is to *bl* as *ba* is to *sa*; it follows that *dc* is to *bl* as *ca* is to *sa*. In other words, the speed at *c* is to that at *b* as *ca* is to *sa*, that is, as the time of fall through *ab*.

The method of measuring the speed of a body along the direction of its fall is thus clear; the speed is assumed to increase directly as the time.

But before we proceed further, since this discussion is to deal with the motion compounded of a uniform horizontal one and one accelerated vertically downwards—the path of a projectile, namely, a parabola—it is necessary that we define some common standard by which we may estimate the velocity, or momentum [*velocitatem, impetum seu momentum*] of both motions; {260} and since from the innumerable uniform velocities one only, and that not selected at random, is to be compounded with a velocity acquired by naturally accelerated motion, I can think of no simpler way of selecting and measuring this than to assume another of the same kind.* For the sake of clearness, draw the vertical line *ac* to meet the horizontal line *bc*. *Ac* is the height and *bc* the amplitude of the semi-parabola *ab*, which is the resultant of the two motions, one that of a body falling [283] from rest at *a*, through the distance *ac*, with naturally accelerated motion, the other a uniform motion along the horizontal *ad*. The speed acquired at *c* by a fall through the distance *ac* is determined by the height *ac*; for the speed of a body falling from the same elevation is always one and the same; but along the horizontal one may give a body an infinite number of uniform speeds. However, in order that I may select one out of this multitude and separate it from the rest in a perfectly definite manner, I will extend the height *ca* upwards to *e* just as far as is necessary and will call this distance *ae* the "sublimity." Imagine a body to fall from rest at *e*; it is clear that we may make its terminal speed at a the same as that with which the same body travels along the horizontal line *ad*; this speed will be such that, in the time of descent along *ea*, it will describe a horizontal distance twice the length of *ea*. This preliminary remark seems necessary.

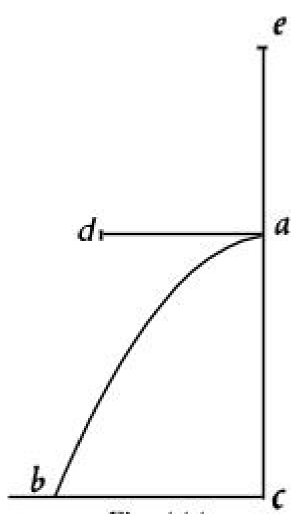

Fig. 111

* Galileo here proposes to employ as a standard of velocity the terminal speed of a body falling freely from a given height. [*Trans.*]

The reader is reminded that above I have called the horizontal line *cb* the "amplitude" of the semi-parabola *ab*; the axis *ac* of this parabola, I have called its "altitude"; but the line *ea* the fall along which determines the horizontal speed I have called the "sublimity." These matters having been explained, I proceed with the demonstration.{261}

SAGR. Allow me, please, to interrupt in order that I may point out the beautiful agreement between this thought of the Author and the views of Plato concerning the origin of the various uniform speeds with which the heavenly bodies revolve. The latter chanced upon the idea that a body could not pass from rest to any given speed and maintain it uniformly except by passing through all the degrees of speed intermediate between the given speed and rest. Plato thought that God, after having created the heavenly bodies, assigned them the proper and uniform speeds with which they were forever to revolve; and that He made them start from rest and move over definite distances under a natural and rectilinear acceleration such as governs the motion of terrestrial bodies. He added that once these bodies had gained their proper and permanent speed, their rectilinear motion was converted into a circular one, the only [284] motion capable of maintaining uniformity, a motion in which the body revolves without either receding from or approaching its desired goal. This conception is truly worthy of Plato; and it is to be all the more highly prized since its underlying principles remained hidden until discovered by our Author who removed from them the mask and poetical dress and set forth the idea in correct historical perspective. In view of the fact that astronomical science furnishes us such complete information concerning the size of the planetary orbits, the distances of these bodies from their centers of revolution, and their velocities, I cannot help thinking that our Author (to whom this idea of Plato was not unknown) had some curiosity to discover whether or not a definite "sublimity" might be assigned to each planet, such that, if it were to start from rest at this particular height and to fall with naturally accelerated motion along a straight line, and were later to change the speed thus acquired into uniform motion, the size of its orbit and its period of revolution would be those actually observed.

SALV. I think I remember his having told me that he once made the computation and found a satisfactory correspondence with observation. But he did not wish to speak of it, lest in {262} view of the odium which his many new discoveries had already brought upon him, this might be adding fuel to the fire. But if any one desires such information he can obtain it for himself from the theory set forth in the present treatment.

We now proceed with the matter in hand, which is to prove:

PROBLEM I, PROPOSITION IV

To determine the momentum of a projectile at each particular point in its given parabolic path.

Let *bec* be the semi-parabola whose amplitude is *cd* and whose height is *db*, which latter extended upwards cuts the tangent of the parabola *ca* in *a*. Through the vertex draw the horizontal line *bi* parallel to *cd*. Now if the amplitude *cd* is equal to the entire height *da*, then *bi* will be equal to *ba* and also to *bd*; and if we take *ab* as the measure of the time required for fall through the distance *ab* and also of the momentum acquired at *b* in consequence of its fall from rest at *a*, then if we turn into a horizontal direction the momentum acquired by fall through *ab* [*impetum ab*] the space traversed in the same interval of time will be represented by *dc* which is twice *bi*. But a body which falls from rest at *b* along the line bd will during the same time-interval fall through the height of the parabola [285] *bd*. Hence a body falling from rest at a, turned into a horizontal direction with the speed *ab* will traverse a space equal to *dc*. Now if one superposes upon this motion a fall along *bd*, traversing the height *bd* while the parabola *bc* is described, then the momentum of the body at the terminal point *c* is the resultant of a uniform horizontal momentum, whose value is represented by *ab*, and of another momentum acquired by fall from *b* to the terminal point *d* or *c*; these two momenta are equal. If, therefore, we take *ab* to be the measure of one of these momenta, say, the uniform horizontal one, then *bi*, which is equal to *bd*, will represent the momentum acquired at *d* or *c*; and *ia* will represent the resultant of these two momenta, that is, the total momentum with which the projectile, travelling along the parabola, strikes at *c*. {263}

With this in mind let us take any point on the parabola, say *e*, and determine the momentum with which the projectile passes that point. Draw the horizontal *ef* and take *bg* a mean proportional between *bd* and *bf*. Now since *ab*, or *bd*, is assumed to be the measure of the time and of the momentum [*momentum velocitatis*] acquired by falling from rest at *b* through the distance *bd*, it follows that bg will measure the time and also the momentum [*impetus*] acquired at *f* by fall from *b*. If therefore we lay off *bo*, equal to *bg*, the diagonal line joining *a* and *o* will represent the momentum at the point *e*; because the length *ab* has been assumed to represent the momentum at *b* which, after diversion into a horizontal direction, remains constant; and because *bo* measures the momentum at *f* or *e*, acquired by fall, from rest at *b*, through the height *bf*. But the square of *ao* equals the sum of the squares of *ab* and *bo*. Hence the theorem sought.

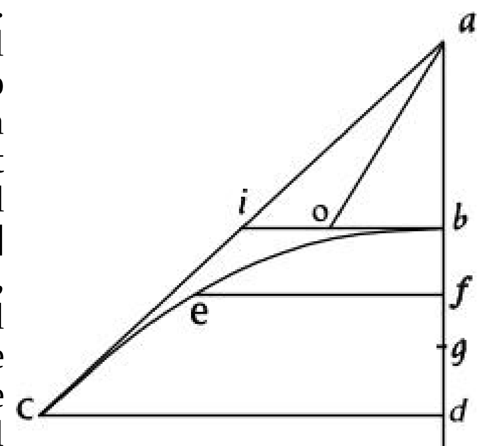

Fig. 112

SAGR. The manner in which you compound these different momenta to obtain their resultant strikes me as so novel that my mind is left in no small confusion. I do not refer to the composition of two uniform motions, even when unequal, and when one takes place along a horizontal, the other along a vertical direction; because in this case I am thoroughly convinced that the resultant is a motion whose square is equal to the sum of the squares of the two components. The confusion arises when one undertakes to compound a uniform horizontal motion with a vertical one which is naturally accelerated. I trust, therefore, we may pursue this discussion more at length.[286]

SIMP. And I need this even more than you since I am not yet as clear in my mind as

I ought to be concerning those fundamental propositions upon which the others rest. Even in the {264} case of the two uniform motions, one horizontal, the other perpendicular, I wish to understand better the manner in which you obtain the resultant from the components. Now, Salviati, you understand what we need and what we desire.

SALV. Your request is altogether reasonable and I will see whether my long consideration of these matters will enable me to make them clear to you. But you must excuse me if in the explanation I repeat many things already said by the Author. Concerning motions and their velocities or momenta [*movimenti e lor velocità o impeti*] whether uniform or naturally accelerated, one cannot speak definitely until he has established a measure for such velocities and also for time. As for time we have the already widely adopted hours, first minutes and second minutes. So for velocities, just as for intervals of time, there is need of a common standard which shall be understood and accepted by everyone, and which shall be the same for all. As has already been stated, the Author considers the velocity of a freely falling body adapted to this purpose, since this velocity increases according to the same law in all parts of the world; thus for instance the speed acquired by a leaden ball of a pound weight starting from rest and falling vertically through the height of, say, a spear's length is the same in all places; it is therefore excellently adapted for representing the momentum [*impeto*] acquired in the case of natural fall.

It still remains for us to discover a method of measuring momentum in the case of uniform motion in such a way that all who discuss the subject will form the same conception of its size and velocity [*grandezza e velocità*]. This will prevent one person from imagining it larger, another smaller, than it really is; so that in the composition of a given uniform motion with one which is accelerated different men may not obtain different values for the resultant. In order to determine and represent such a momentum and particular speed [*impeto e velocità particolare*] our Author has found no better method than to use the momentum acquired by a body in naturally accelerated motion.

[287]

The speed of a body which has in this manner acquired any {265} momentum whatever will, when converted into uniform motion, retain precisely such a speed as, during a time-interval equal to that of the fall, will carry the body through a distance equal to twice that of the fall. But since this matter is one which is fundamental in our discussion it is well that we make it perfectly clear by means of some particular example.

Let us consider the speed and momentum acquired by a body falling through the height, say, of a spear [*picca*] as a standard which we may use in the measurement of other speeds and momenta as occasion demands; assume for instance that the time of such a fall is four seconds [*minuti secondi d'ora*]; now in order to measure the speed acquired from a fall through any other height, whether greater or less, one must not conclude that these speeds bear to one another the same ratio as the heights of fall; for instance, it is not true that a fall through four times a given height confers a speed four times as great as that acquired by descent through the given height; because the speed of a naturally accelerated motion does not vary in proportion to the time. As has been shown above, the ratio of the spaces is equal to the square of the ratio of the times.

If, then, as is often done for the sake of brevity, we take the same limited straight line as the measure of the speed, and of the time, and also of the space traversed during that time, it follows that the duration of fall and the speed acquired by the same body in passing over any other distance, is not represented by this second distance, but by a mean proportional between the two distances. This I can better illustrate by an example. In the vertical line ac, lay off the portion ab to represent the distance traversed by a body falling freely with accelerated motion: the time of fall may be represented by any limited straight line, but for the sake of brevity, we shall represent it by the same length ab; this length may also be employed as a measure of the momentum and speed acquired during the motion; in short, let ab be a measure of the various physical quantities which enter this discussion.

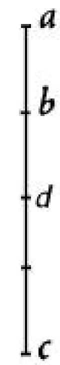

Fig. 113

Having agreed arbitrarily upon ab as a measure of these {266} three different quantities, namely, space, time, and momentum, our next task is to find the time required for fall through a [288] given vertical distance ac, also the momentum acquired at the terminal point *c*, both of which are to be expressed in terms of the time and momentum represented by *ab*. These two required quantities are obtained by laying off ad, a mean proportional between *ab* and *ac*; in other words, the time of fall from *a* to c is represented by ad on the same scale on which we agreed that the time of fall from a to b should be represented by ab. In like manner we may say that the momentum [*impeto o grado di velocità*] acquired at c is related to that acquired at b, in the same manner that the line ad is related to ab, since the velocity varies directly as the time, a conclusion, which although employed as a postulate in Proposition III, is here amplified by the Author.

This point being clear and well-established we pass to the consideration of the momentum [*impeto*] in the case of two compound motions, one of which is compounded of a uniform horizontal and a uniform vertical motion, while the other is compounded of a uniform horizontal and a naturally accelerated vertical motion. If both components are uniform, and one at right angles to the other, we have already seen that the square of the resultant is obtained by adding the squares of the components [p. 257] as will be clear from the following illustration.

Let us imagine a body to move along the vertical *ab* with a uniform momentum [*impeto*] of 3, and on reaching *b* to move toward *c* with a momentum [*velocità ed impeto*] of 4, so that during the same time-interval it will traverse 3 cubits along the vertical and 4 along the horizontal. But a particle which moves with the resultant velocity [*velocità*] will, in the same time, traverse the diagonal *ac*, whose length is not 7 cubits—the sum of *ab* (3) and *bc* (4)—but 5, which is in potenza equal to the sum of 3 and 4, that is, the squares of 3 and 4 when added make 25, which is the square of *ac*, and is equal to the sum of the squares {267} of *ab* and *bc*. Hence *ac* is represented by the side—or we may say the root—of a square whose area is 25, namely 5.

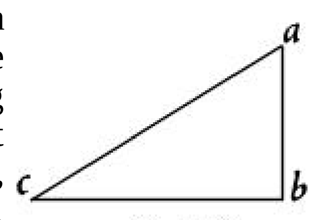

Fig. 114

As a fixed and certain rule for obtaining the momentum which [289] results from two uniform momenta, one vertical, the other horizontal, we have therefore the following: take the square of each, add these together, and extract the square root of the sum, which will be the momentum resulting from the two. Thus, in the above example, the body which in virtue of its vertical motion would strike the horizontal plane with a momentum [*forza*] of 3, would owing to its horizontal motion alone strike at c with a momentum of 4; but if the body strikes with a momentum which is the resultant of these two, its blow will be that of a body moving with a momentum [*velocità e forza*] of 5; and such a blow will be the same at all points of the diagonal ac, since its components are always the same and never increase or diminish.

Let us now pass to the consideration of a uniform horizontal motion compounded with the vertical motion of a freely falling body starting from rest. It is at once clear that the diagonal which represents the motion compounded of these two is not a straight line, but, as has been demonstrated, a semi-parabola, in which the momentum [*impeto*] is always increasing because the speed [*velocità*] of the vertical component is always increasing. Wherefore, to determine the momentum [*impeto*] at any given point in the parabolic diagonal, it is necessary first to fix upon the uniform horizontal momentum [*impeto*] and then, treating the body as one falling freely, to find the vertical momentum at the given point; this latter can be determined only by taking into account the duration of fall, a consideration which does not enter into the composition of two uniform motions where the velocities and momenta are always the same; but here where one of the component motions has an initial value of zero and increases its speed [*velocità*] in direct proportion to the time, it follows that the time must determine the speed [*velocità*] at the assigned point. It only remains to obtain the momentum resulting from these two components (as in the case of uniform motions) by placing the square of the resultant equal {268} to the sum of the squares of the two components. But here again it is better to illustrate by means of an example.

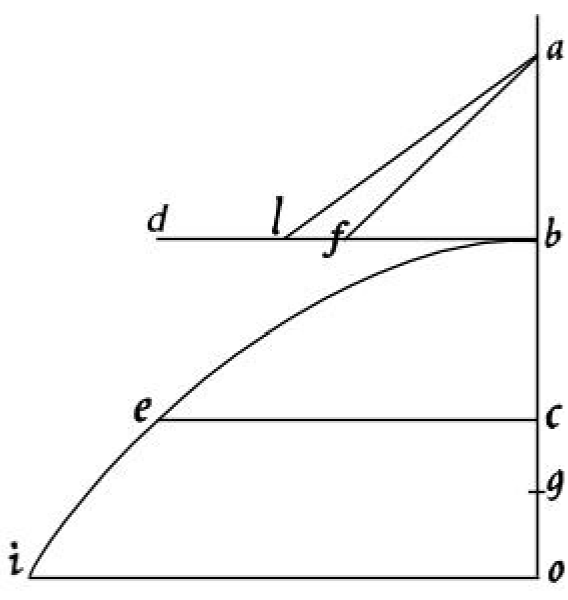

Fig. 115

On the vertical *ac* lay off any portion *ab* which we shall employ as a measure of the space traversed by a body falling freely along the perpendicular, likewise as a measure of the time and also of the speed [*grado di velocità*] or, we may say, of the momenta [*impeti*].

It is at once clear that if the momentum of a [290] body at *b*, after having fallen from rest at *a*, be diverted along the horizontal direction bd, with uniform motion, its speed will be such that, during the time-interval *ab*, it will traverse a distance which is represented by the line *bd* and which is twice as great as *ab*. Now choose a point *c*, such that *bc* shall be equal to *ab*, and through *c* draw the line *ce* equal and parallel to *bd*; through the points *b* and *e* draw the parabola *bei*. And since, during the time- interval *ab*, the horizontal distance bd or *ce*, double the length *ab*, is traversed with the momentum *ab*, and since during an equal time-interval the vertical distance *bc* is traversed, the body acquiring at *c* a momentum represented by the same horizontal, *bd*, it follows that during the time *ab* the body will pass from *b* to *e* along the parabola *be*, and will reach *e* with a momentum compounded of two momenta each equal to *ab*. And since one of these is horizontal and the other vertical, the square of the resultant momentum is equal to the sum of the squares of these two components, *i.e.*, equal to twice either one of them.

Therefore, if we lay off the distance *bf*, equal to *ba*, and draw the diagonal *af*, it follows that the momentum [*impeto e percossa*] at *e* will exceed that of a body at *b* after having fallen from {269} *a*, or what is the same thing, will exceed the horizontal momentum [*percossa dell'impeto*] along *bd*, in the ratio of *af* to *ab*.

Suppose now we choose for the height of fall a distance *bo* which is not equal to but greater than *ab*, and suppose that *bg* represents a mean proportional between *ba* and *bo*; then, still retaining *ba* as a measure of the distance fallen through, from rest at *a*, to *b*, also as a measure of the time and of the momentum which the falling body acquires at *b*, it follows that *bg* will be the measure of the time and also of the momentum which the body acquires in falling from *b* to *o*. Likewise just as the momentum *ab* during the time *ab* carried the body a distance along the horizontal equal to twice *ab*, so now, during the time-interval *bg*, the body will be carried in a horizontal direction through a distance which is greater in the ratio of *bg* to *ba*. Lay off *lb* equal to *bg* and draw the diagonal *al*, from which we have a quantity compounded of two velocities [*impeti*] one horizontal, the other vertical; these determine the parabola. The horizontal and uniform velocity is that acquired at *b* in falling from *a*; the other is that acquired at *o*, or, we may say, at *l*, by a body falling through the distance *bo*, during a time measured by the line *bg*, [291] which line *bg* also represents the momentum of the body. And in like manner we may, by taking a mean proportional between the two heights, determine the momentum [*impeto*] at the extreme end of the parabola where the height is less than the sublimity *ab*; this mean proportional is to be drawn along the horizontal in place of *bf*, and also another diagonal in place of *af*, which diagonal will represent the momentum at the extreme end of the parabola.

To what has hitherto been said concerning the momenta, blows or shocks of projectiles, we must add another very important consideration; to determine the force and energy of the shock [*forza ed energia della percossa*] it is not sufficient to consider only the speed of the projectiles, but we must also take into account the nature and condition of the target which, in no small degree, determines the efficiency of the blow. First of all it is well known that the target suffers violence from the speed {270} [*velocità*] of the projectile in proportion as it partly or entirely stops the motion; because if the blow falls upon an object which yields to the impulse [*velocità del percuziente*] without resistance such a blow will be of no effect; likewise when one attacks his enemy with a spear and overtakes him at an instant when he is fleeing with equal speed there will be no blow but merely a harmless touch. But if the shock falls upon an object which yields only in part then the blow will not have its full effect, but the damage will be in proportion to the excess of the speed of the projectile over that of the receding body; thus, for example, if the shot reaches the target with a speed of 10 while the latter recedes with a speed of 4, the momentum and shock [*impeto e percossa*] will be represented by 6. Finally the blow will be a maximum, in so far as the projectile is concerned, when the target does not recede at all but if possible completely resists and stops the motion of the projectile. I have said in so far as the projectile is concerned because if the target should approach the projectile the shock of collision [*colpo e l'incontro*] would be greater in proportion as the sum of the two speeds is greater than that of the projectile alone.

Moreover it is to be observed that the amount of yielding in the target depends not only upon the quality of the material, as regards hardness, whether it be of iron, lead, wool, etc., but [292] also upon its position. If the position is such that the shot strikes it at right angles, the momentum imparted by the blow [*impeto del colpo*] will be a maximum; but if the motion be oblique, that is to say slanting, the blow will be weaker; and more and more so in proportion to the obliquity; for, no matter how hard the material of the target thus situated, the entire momentum [*impeto e moto*] of the shot will not be spent and stopped; the projectile will slide by and will, to some extent, continue its motion along the surface of the opposing body.

All that has been said above concerning the amount of momentum in the projectile at the extremity of the parabola must be understood to refer to a blow received on a line at right angles to this parabola or along the tangent to the parabola at the given {271} point; for, even though the motion has two components, one horizontal, the other vertical, neither will the momentum along the horizontal nor that upon a plane perpendicular to the horizontal be a maximum, since each of these will be received obliquely.

SAGR. Your having mentioned these blows and shocks recalls to my mind a problem, or rather a question, in mechanics of which no author has given a solution or said anything which diminishes my astonishment or even partly relieves my mind.

My difficulty and surprise consist in not being able to see whence and upon what principle is derived the energy and immense force [*energia e forza immensa*] which makes its appearance in a blow; for instance we see the simple blow of a hammer, weighing not more than 8 or 10 lbs., overcoming resistances which, without a blow, would not yield to the weight of a body producing impetus by pressure alone, even though that body weighed many hundreds of pounds. I would like to discover a method of measuring the force [*forza*] of such a percussion. I can hardly think it infinite, but incline rather to the view that it has its limit and can be counterbalanced and measured by other forces, such as weights, or by levers or screws or other mechanical instruments which are used to multiply forces in a manner which I satisfactorily understand.

SALV. You are not alone in your surprise at this effect or in obscurity as to the cause of this remarkable property. I studied this matter myself for a while in vain; but my confusion merely increased until finally meeting our Academician I received from [293] him great consolation. First he told me that he also had for a long time been groping in the dark; but later he said that, after having spent some thousands of hours in speculating and contemplating thereon, he had arrived at some notions which are far removed from our earlier ideas and which are remarkable for their novelty. First he told me that he also had for a long time been groping in the dark; but later he said that, after having spent some thousands of hours in speculating and contemplating thereon, he had arrived at some notions which are far removed from our earlier ideas and which are remarkable for their novelty. And since now I know that you would gladly hear what these novel ideas are I shall not wait for you to ask but promise that, as soon as our discussion of projectiles is completed, I will explain all these fantasies, or if you please, {272} vagaries, as far as I can recall them from the words of our Academician. In the meantime we proceed with the propositions of the author.

PROPOSITION V, PROBLEM

Having given a parabola, find the point, in its axis extended upwards, from which a particle must fall in order to describe this same parabola.

Let *ab* be the given parabola, *hb* its amplitude, and he its axis extended. The problem is to find the point *e* from which a body must fall in order that, after the momentum which it acquires at *a* has been diverted into a horizontal direction, it will describe the parabola *ab*. Draw the horizontal *ag*, parallel to *bh*, and having laid off *af* equal to *ah*, draw the straight line *bf* which will be a tangent to the parabola at *b*, and will intersect the horizontal *ag* at *g*: choose *e* such that *ag* will be a mean proportional between *af* and *ae*. Now I say that *e* is the point above sought. That is, if a body falls from rest at this point *e*, and if the momentum acquired at the point *a* be diverted into a horizontal direction, and compounded with the momentum acquired at h in falling from rest at *a*, then the body will describe the parabola *ab*. For if we understand *ea* to be the measure of the time of fall from *e* to *a*, and also of the momentum acquired at *a*, then *ag* (which is a mean proportional between *ea* and *af*) will represent the time and momentum of fall from *f* to *a* or, what is the same thing, from a to h; and since a body falling from *e*, during the time *ea*, will, owing to the momentum acquired at *a*, traverse at uniform speed a horizontal distance which is twice *ea*, it follows that, the body will if impelled by the same momentum, during the time-interval *ag* traverse a distance equal to twice *ag* which is the half of *bh*. This is true because,{273} in the case of uniform motion, the spaces traversed vary directly as the times. And likewise if the motion be vertical and start from rest, the body will describe the distance ah in the [294] time *ag*. Hence the amplitude *bh* and the altitude ah are traversed by a body in the same time. Therefore the parabola *ab* will be described by a body falling from the sublimity of *e*. Q. E. F.

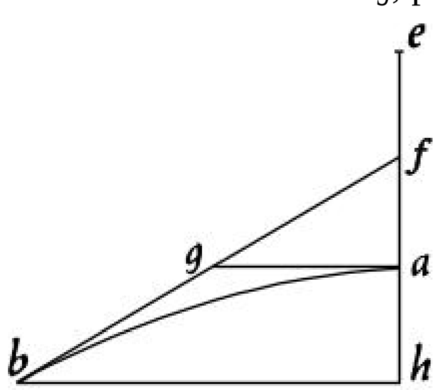

Fig. 116

COROLLARY

Hence it follows that half the base, or amplitude, of the semi-parabola (which is one-quarter of the entire amplitude) is a mean proportional between its altitude and the sublimity from which a falling body will describe this same parabola.

PROPOSITION VI, PROBLEM

Given the sublimity and the altitude of a parabola, to find its amplitude.

Let the line *ac*, in which lie the given altitude *cb* and sublimity *ab*, be perpendicular to the horizontal line *cd*. The problem is to find the amplitude, along the horizontal cd, of the semi-parabola which is described with the sublimity ba and altitude bc. Lay off cd equal to twice the mean proportional between cb and ba. Then cd will be the amplitude sought, as is evident from the preceding proposition.

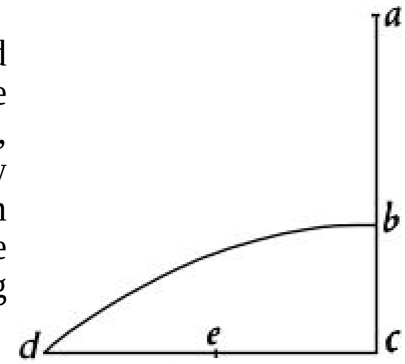

Fig. 117

THEOREM. PROPOSITION VII

If projectiles describe semi-parabolas of the same amplitude, the momentum required to describe that one whose amplitude is double its altitude is less than that required for any other.{274}

Let *bd* be a semi-parabola whose amplitude *cd* is double its altitude *cb*; on its axis extended upwards lay off *ba* equal to its altitude *bc*. Draw the line *ad* which will be a tangent to the parabola at *d* and will cut the horizontal line *be* at the point *e*, making *be* equal to *bc* and also to *ba*. It is evident that this parabola will be described by a projectile whose uniform horizontal momentum is that which it would acquire at *b* in falling from

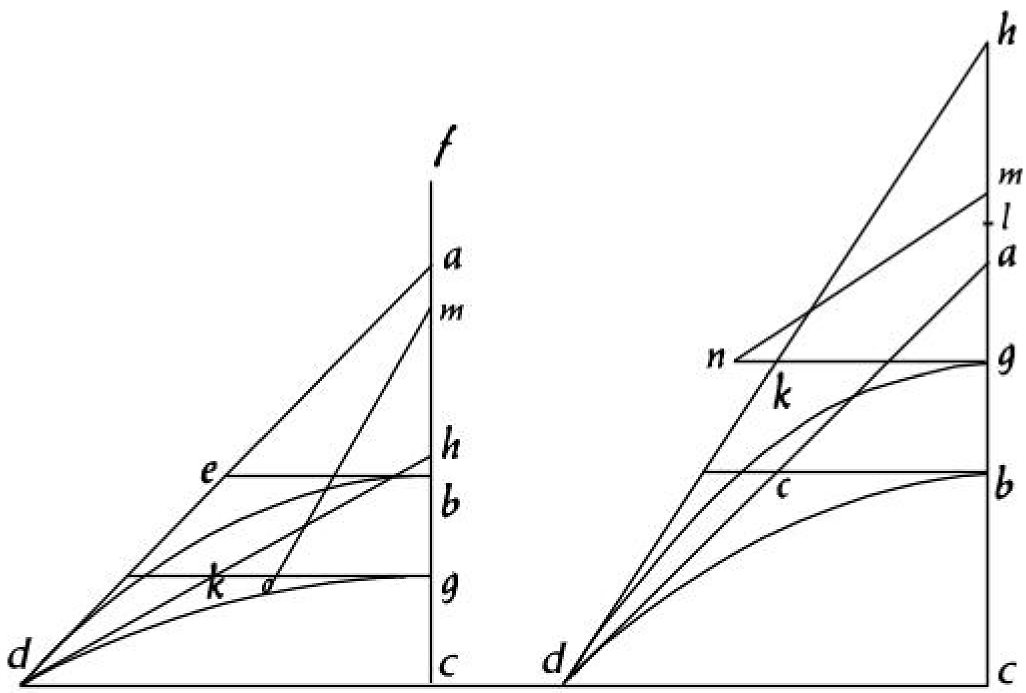

Fig. 118

rest at *a* and whose naturally accelerated vertical momentum is that of the body falling to *c*, from rest at *b*. From this it follows that the momentum at the terminal point *d*, compounded of these two, is represented by the diagonal *ae*, whose square is equal to the sum of the squares of the two components. Now let *gd* be any other parabola whatever having the same amplitude *cd*, but whose altitude *cg* is either greater or less than the altitude *bc*. Let *hd* be the tangent cutting the horizontal [295] through *g* at *k*. Select a point *l* such that $hg:gk = gk:gl$. Then from a preceding proposition [V], it follows that *gl* will be the {275} height from which a body must fall in order to describe the parabola *gd*.

Let *gm* be a mean proportional between *ab* and *gl*; then *gm* will [Prop. IV] represent the time and momentum acquired at *g* by a fall from *l*; for *ab* has been assumed as a measure of both time and momentum. Again let *gn* be a mean proportional between *bc* and *cg*; it will then represent the time and momentum which the body acquires at *c* in falling from *g*. If now we join *m* and *n*, this line *mn* will represent the momentum at *d* of the projectile traversing the parabola *dg*, which momentum is, I say, greater than that of the projectile travelling along the parabola *bd* whose measure was given by *ae*. For since *gn* has been taken as a mean proportional between *bc* and *gc*; and since *bc* is equal to *be* and also to *kg* (each of them being the half of *dc*) it follows that $cg:gn = gn:gk$, and as *cg* or (*hg*) is to *gk* so is ng^2 to gk^2: but by construction $hg:gk = gk:gl$. Hence $ng^2:gk^2 = gk:gl$. But $gk:gl = gk^2:gm^2$, since *gm* is a mean proportional between *kg* and *gl*. Therefore the three squares *ng, kg, mg* form a continued proportion, $gn^2:gk^2 = gk^2:gm^2$. And the sum of the two extremes which is equal to the square of *mn* is greater than twice the square of *gk*; but the square of *ae* is double the square of *gk*. Hence the square of *mn* is greater than the square of *ae* and the length *mn* is greater than the length *ae*. Q. E. D.

[296]

COROLLARY

Conversely it is evident that less momentum will be required to send a projectile from the terminal point d along the parabola *bd* than along any other parabola having an elevation greater or less than that of the parabola *bd*, for which the tangent at *d* makes an angle of 45° with the horizontal. From which it follows that if projectiles are fired from the terminal point *d*, all having the same speed, but each having a different elevation, the maximum range, *i.e.*, amplitude of the semi-parabola or of the entire parabola, will be obtained when the elevation is 45°: the {276} other shots, fired at angles greater or less will have a shorter range.

SAGR. The force of rigid demonstrations such as occur only in mathematics fills me with wonder and delight. From accounts given by gunners, I was already aware of the fact that in the use of cannon and mortars, the maximum range, that is the one in which the shot goes farthest, is obtained when the elevation is 45° or, as they say, at the sixth point of the quadrant; but to understand why this happens far outweighs the mere information obtained by the testimony of others or even by repeated experiment.

SALV. What you say is very true. The knowledge of a single fact acquired through a discovery of its causes prepares the mind to understand and ascertain other facts without need of recourse to experiment, precisely as in the present case, where by argumentation alone the Author proves with certainty that the maximum range occurs when the elevation is 45°. He thus demonstrates what has perhaps never been observed in experience, namely, that of other shots those which exceed or fall short of 45° by equal amounts have equal ranges; so that if the balls have been fired one at an elevation of 7 points, the other at 5, they will strike the level at the same distance: the same is true if the shots are fired at 8 and at 4 points, at 9 and at 3, etc. Now let us hear the demonstration of this.

[297]

THEOREM. PROPOSITION VIII

> The amplitudes of two parabolas described by projectiles fired with the same speed, but at angles of elevation which exceed and fall short of 45° by equal amounts, are equal to each other.

In the triangle *mcb* let the horizontal side bc and the vertical *cm*, which form a right angle at *c*, be equal to each other; then the angle *mbc* will be a semi-right angle; let the line *cm* be prolonged to *d*, such a point that the two angles at b, namely *mbe* and *mbd*, one above and the other below the diagonal *mb*, shall be equal. It is now to be proved that in the case of two parabolas {277} described by two projectiles fired from *b* with the same speed, one at the angle of *ebc*, the other at the angle of *dbc*, their amplitudes will be equal. Now since the external angle *bmc* is equal to the sum of the internal angles *mdb* and *dbm* we may also equate to them the angle *mbc*; but if we replace the angle *dbm* by *mbe*, then this same angle mbc is equal to the two *mbe* and *bdc*: and if we subtract from each side of this equation the angle *mbe*, we have the remainder *bdc* equal to the remainder *ebc*. Hence the two triangles *dcb* and *bce* are similar. Bisect the straight lines *dc* and *ec* in the points *h* and *f*: and draw the lines *hi* and *fg* parallel to the horizontal *cb*, and choose l such that *dh:hi = ih:hl*. Then the triangle *ihl* will be similar to *ihd*, and also to the *egf*; and since *ih* and *gf* are equal, each being half of *bc*, it follows that *hl* is equal to *fe* and also to *fc*; and if we add to each of these the common part *fh*, it will be seen that *ch* is equal to *fl*.

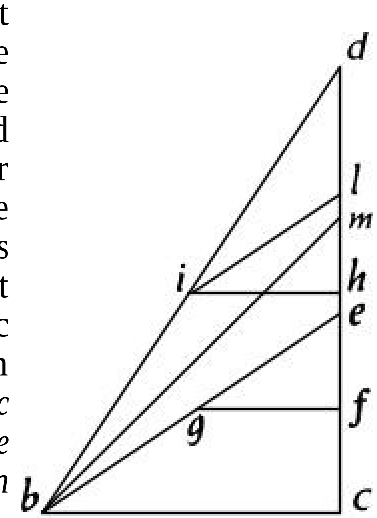

Fig. 119

Let us now imagine a parabola described through the points *h* and *b* whose altitude is *hc* and sublimity *hl*. Its amplitude will be *cb* which is double the length *hi* since *hi* is a mean proportional between *dh* (or *ch*) and *hl*. The line *db* is tangent to the parabola at b, since *ch* is equal to *hd*. If again we imagine a parabola described through the points *f* and *b*, with a sublimity *fl* and altitude *fc*, of which the mean proportional is *fg*, or one-half of *cb*, then, as before, will *cb* be the amplitude and the line *eb* a tangent at *b*; for *ef* and *fc* are equal.

[298]

But the two angles *dbc* and *ebc*, the angles of elevation, differ by equal amounts from a 45° angle. Hence follows the proposition.

THEOREM. PROPOSITION IX

The amplitudes of two parabolas are equal when their altitudes and sublimities are inversely proportional.{278}

Let the altitude *gf* of the parabola *fh* bear to the altitude *cb* of the parabola *bd* the same ratio which the sublimity ba bears to the sublimity *fe*, then I say the amplitude *hg* is equal to the amplitude dc. For since the first of these quantities, *gf*, bears to the second *cb* the same ratio which the third, *ba*, bears to the fourth *fe*, it follows that the area of the rectangle *gf.fe* is equal to that of the rectangle *cb.ba*; therefore squares which are equal to these rectangles are equal to each other. But [by Proposition VI] the square of half of *gh* is equal to the rectangle *gf.fe*; and the square of half of cd is equal to the rectangle *cb.ba*. Therefore these squares and their sides and the doubles of their sides are equal. But these last are the amplitudes *gh* and *cd*. Hence follows the proposition.

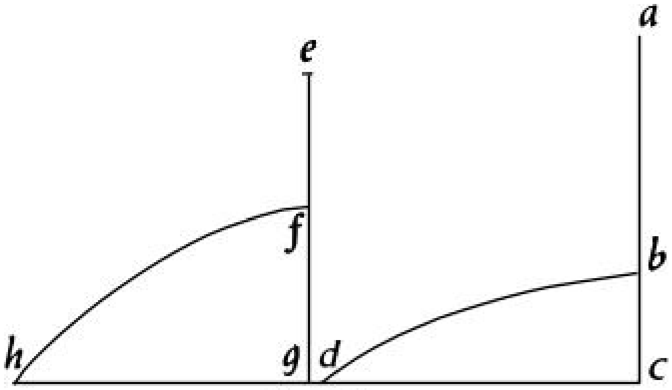

Fig. 120

LEMMA FOR THE FOLLOWING PROPOSITION

If a straight line be cut at any point whatever and mean proportionals between this line and each of its parts be taken, the sum of the squares of these mean proportionals is equal to the square of the entire line.

Let the line *ab* be cut at *c*. Then I say that the square of the mean proportional between *ab* and *ac* plus the square of the mean proportional between *ab* and *cb* is equal to the square of the whole line *ab*. This is evident as soon as we describe a semicircle upon the entire line *ab*, erect a perpendicular cd at *c*, and draw *da* and *db*. For *da* is a mean proportional between *ab* and *ac* while [299] *db* is a mean proportional between *ab* and *bc*: and since the angle *adb*, inscribed in a semicircle, is a right angle the sum of {279} the squares of the lines *da* and *db* is equal to the square of the entire line *ab*. Hence follows the proposition.

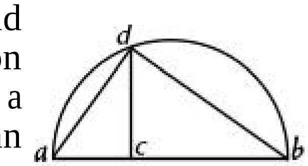

Fig. 121

THEOREM. PROPOSITION X

The momentum [*impetus seu momentum*] acquired by a particle at the terminal point of any semi-parabola is equal to that which it would acquire in falling through a vertical distance equal to the sum of the sublimity and the altitude of the semi-parabola.*

* In modern mechanics this well-known theorem assumes the following from: *The speed of a projectile at any point is that produced by a fall from the directrix.* [*Trans.*]

Let *ab* be a semi-parabola having a sublimity *da* and an altitude *ac*, the sum of which is the perpendicular *dc*. Now I say the momentum of the particle at *b* is the same as that which it would acquire in falling freely from *d* to *c*. Let us take the length of *dc* itself as a measure of time and momentum, and lay off *cf* equal to the mean proportional between *cd* and *da*; also lay off *ce* a mean proportional between *cd* and *ca*. Now *cf* is the measure of the time and of the momentum acquired by fall, from rest at *d*, through the distance *da*; while *ce* is the time and momentum of fall, from rest at *a*, through the distance *ca*; also the diagonal, *ef* will represent a momentum which is the resultant of these two, and is therefore the momentum at the terminal point of the parabola, *b*.

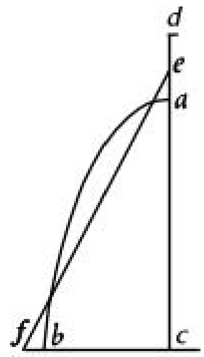

Fig. 122

And since *dc* has been cut at some point a and since *cf* and *ce* are mean proportionals between the whole of *cd* and its parts, *da* and *ac*, it follows, from the preceding lemma, that the sum of the squares of these mean proportionals is equal to the square of the whole: but the square of *ef* is also equal to the sum of these same squares; whence it follows that the line *ef* is equal to *dc*.

Accordingly the momentum acquired at *c* by *a* particle in falling from *d* is the same as that acquired at *b* by a particle traversing the parabola *ab*. Q. E. D.

{280}

COROLLARY

Hence it follows that, in the case of all parabolas where the sum of the sublimity and altitude is a constant, the momentum at the terminal point is a constant.

PROBLEM. PROPOSITION XI

Given the amplitude and the speed [*impetus*] at the terminal point of a semi-parabola, to find its amplitude.

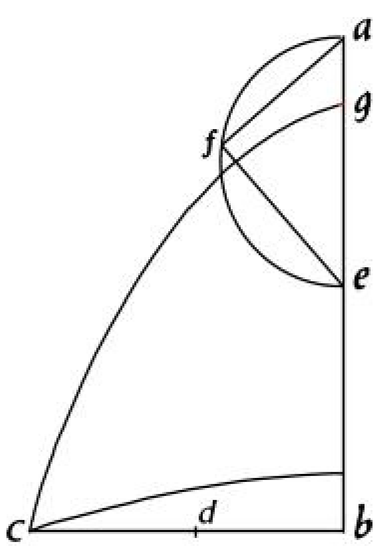

Fig. 123

Let the given speed be represented by the vertical line *ab*, and the amplitude by the horizontal line *bc*; it is required to find the sublimity of the semi-parabola whose terminal speed is *ab* and amplitude *bc*. From what precedes [Cor. Prop. V] it is clear that half the amplitude *bc* is a mean proportional between [300]the altitude and sublimity of the parabola of which the terminal speed is equal, in accordance with the preceding proposition, to the speed acquired by a body in falling from rest at a through the distance *ab*. Therefore the line *ba* must be cut at a point such that the rectangle formed by its two parts will be equal to the square of half *bc*, namely *bd*. Necessarily, therefore, *bd* must not exceed the half of *ba*; for of all the rectangles formed by parts of a straight line the one of greatest area is obtained when the line is divided into two equal parts. Let *e* be the middle point of the line *ab*; and now if *bd* be equal to be the problem is solved; for *be* will be the altitude and *ea* the sublimity of the parabola. (Incidentally we may observe a consequence already demonstrated, namely: of all parabolas described with any given terminal speed that for which the elevation is 45° will have the maximum amplitude.)

But suppose that *bd* is less than half of *ba* which is to be {281} divided in such a way that the rectangle upon its parts may be equal to the square of *bd*. Upon *ea* as diameter describe a semi-circle *efa*, in which draw the chord *af*, equal to *bd*: join *fe* and lay off the distance *eg* equal to *fe*. Then the rectangle *bg.g*a plus the square of *eg* will be equal to the square of *ea*, and hence also to the sum of the squares of *af* and *fe*. If now we subtract the equal squares of *fe* and *ge* there remains the rectangle *bg.ga* equal to the square of *af*, that is, of *bd*, a line which is a mean proportional between *bg* and *ga*; from which it is evident that the semi-parabola whose amplitude is *bc* and whose terminal speed [*impetus*] is represented by *ba* has an altitude *bg* and a sublimity *ga*.

If however we lay off *bi* equal to *ga*, then *bi* will be the altitude of the semi-parabola *ic*, and *ia* will be its sublimity. From the preceding demonstration we are able to solve the following problem.

PROBLEM. PROPOSITION XII

To compute and tabulate the amplitudes of all semi-parabolas which are described by projectiles fired with the same initial speed [*impetus*].

From the foregoing it follows that, whenever the sum of the altitude and sublimity is a constant vertical height for any set of parabolas, these parabolas are described by projectiles having the same initial speed; all vertical heights thus [301] obtained are therefore included between two parallel horizontal lines. Let *cb* represent a horizontal line and *ab* a vertical line of equal length; draw the diagonal *ac*; the angle *acb* will be one of 45°; let *d* be the middle point of the vertical line *ab*. Then the semi-parabola *dc* is the one which is determined by the sublimity *ad* and the altitude *db*, while its terminal speed at *c* is that which would be acquired at *b* by a particle falling from rest at *a*. If now *ag* be drawn parallel to bc, the sum of the altitude and sublimity for any other semi-parabola having the same terminal speed will, in the manner explained, be equal to the distance between the parallel lines *ag*, and *bc*. Moreover, since {282} it has already been shown that the amplitudes of two semi-parabolas are the same when their angles of elevation differ from 45° by like amounts it follows that the same computation which is employed for the larger elevation will serve also for the smaller. Let us

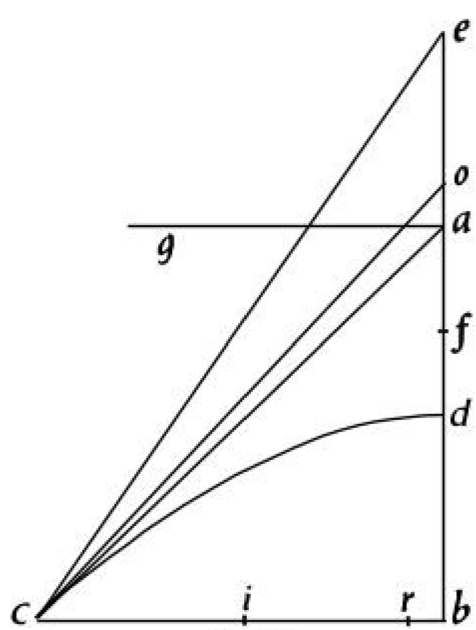

Fig. 124

also assume 10000 as the greatest amplitude for a parabola whose angle of elevation is 45°; this then will be the length of the line *ba* and the amplitude of the semi-parabola *bc*. This number, 10000, is selected because in these calculations we employ a table of tangents in which this is the value of the tangent of 45°. And now, coming down to business, draw the straight line *ce* making an acute angle *ecb* greater than *acb*: the problem now is to draw the semi-parabola to which the line *ec* is a tangent and for which the sum of the sublimity and the altitude is the distance *ba*. Take the length of the tangent* *be* from the table of tangents, using the angle *bce* as an argument: let *f* be the middle point of *be* ; next find a third proportional to *bf* and *bi* (the half of *bc*), which is of necessity greater than *fa*.** Call this *fo*. We have now discovered that, for the parabola inscribed [302] in the triangle *ecb* having the tangent *ce* and the amplitude *cb*, the altitude is *bf* and the sublimity *fo*. But the total length of *bo* exceeds the distance between the parallels *ag* and *cb*, while our problem was to keep it equal to this distance: for both the parabola sought and the parabola *dc* are described {283} by projectiles fired from *c* with the same speed. Now since an infinite number of greater and smaller parabolas, similar to each other, may be described within the angle *bce* we must find another parabola which like *cd* has for the sum of its altitude and sublimity the height *ba*, equal to *bc*.

Therefore lay off *cf* so that, *ob:ba* = *bc:cr*; then *cr* will be the amplitude of a semi-parabola for which *bce* is the angle of elevation and for which the sum of the altitude and sublimity is the distance between the parallels *ga* and *cb*, as desired. The process is therefore as follows: One draws the tangent of the given angle *bce*; takes half of this tangent, and adds to it the quantity, *fo*, which is a third proportional to the half of this tangent and the half of *bc*; the desired amplitude *cr* is then found from the following proportion *ob:ba* = *bc:cr*. For example let the angle *ecb* be one of 50° its tangent is 11918, half of which, namely *bf*, is 5959; half of *bc* is 5000; the third proportional of these halves is 4195, which added to *bf* gives the value 10154 for *bo*. Further, as *ob* is to *ab*, that is, as 10154 is to 10000, so is *bc*, or 10000 (each being the tangent of 45°) to *cr*, which is the amplitude sought and which has the value 9848, the maximum amplitude being *bc*, or 10000. The amplitudes of the entire parabolas are double these, namely, 19696 and 20000. This is also the amplitude of a parabola whose angle of elevation is 40°, since it deviates by an equal amount from one of 45°.

[303]

SAGR. In order to thoroughly understand this demonstration I need to be shown how the third proportional of *bf* and *bi* is, as the Author indicates, necessarily greater than *fa*.

SALV. This result can, I think, be obtained as follows. The square of the mean proportional between two lines is equal to the rectangle formed by these two lines. Therefore the square of *bi* (or of *bd* which is equal to *be*) must be equal to the rectangle formed by *fb* and the desired third proportional. This third proportional is necessarily

* The reader will observe that the word "tangent" is here used in sense somewhat different from that of the preceding sentence. The "tangent *ec*" is a line which touches the parabola at *c* ; but the "tangent *eb*" is the side of the right-angled triangle which lies opposite the angle *ecb*, a line whose length is proportional to the numerical value of the tangent of this angle. [*Trans.*]

** This fact is demonstrated in the third paragraph below, when laid off above the point *f* it extends beyond the parallel *ag*.

greater than *fa* because the rectangle formed by *bf* and *fa* is less than the square of *bd* by an amount equal to the square of *df*, as shown in Euclid, II.I. Besides it is to be observed that the point *f*, which is the middle point of the {284} tangent *eb*, falls in general above *a* and only once at *a*; in which cases it is self-evident that the third proportional to the half of the tangent and to the sublimity be lies wholly above *a*. But the Author has taken a case where it is not evident that the third proportional is always greater than *fa*, so that

Now let us proceed. It will be worth while, by the use of this table, to compute another giving the altitudes of these semi-parabolas described by projectiles having the same initial speed. The construction is as follows: [304]

Amplitudes of semi-parabolas described with the same initial speed

Angle of Elevation°		Angle of Elevation°
45	10000	
46	9994	44
47	9976	43
48	9945	42
49	9902	41
50	9848	40
51	9782	39
52	9704	38
53	9612	37
54	9511	36
55	9396	35
56	9272	34
57	9136	33
58	8989	32
59	8829	31
60	8659	30
61	8481	29
62	8290	28
63	8090	27
64	7880	26
65	7660	25
66	7431	24
67	7191	23
68	6944	22
69	6692	21
70	6428	20
71	6157	19
72	5878	18
73	5592	17
74	5300	16
75	5000	15
76	4694	14
77	4383	13

Altitudes of semi-parabolas described with the same initial speed

Angle of Elevation°		Angle of Elevation°	
1	3	46	5173
2	13	47	5346
3	28	48	5523
4	50	49	5698
5	76	50	5868
6	108	51	6038
7	150	52	6207
8	194	53	6379
9	245	54	6546
10	302	55	6710
11	365	56	6873
12	432	57	7033
13	506	58	7190
14	585	59	7348
15	670	60	7502
16	760	61	7649
17	855	62	7796
18	955	63	7939
19	1060	64	8078
20	1170	65	8214
21	1285	66	8346
22	1402	67	8474
23	1527	68	8597
24	1685	69	8715
25	1786	70	8830
26	1922	71	8940
27	2061	72	9045
28	2204	73	9144
29	2351	74	9240
30	2499	75	9330
31	2653	76	9415
32	2810	77	9493
33	2967	78	9567

Amplitudes of semi-parabolas described with the same initial speed

Angle of Elevation°		Angle of Elevation°
78	4067	12
79	3746	11
80	3420	10
81	3090	9
82	2756	8
83	2419	7
84	2079	6
85	1736	5
86	1391	4
87	1044	3
88	698	2
89	349	1

Altitudes of semi-parabolas described with the same initial speed

Angle of Elevation°		Angle of Elevation°	
34	3128	79	9636
35	3289	80	9698
36	3456	81	9755
37	3621	82	9806
38	3793	83	9851
39	3962	84	9890
40	4132	85	9924
41	4302	86	9951
42	4477	87	9972
43	4654	88	9987
44	4827	89	9998
45	5000	90	10000

[305]

PROBLEM. PROPOSITION XIII

From the amplitudes of semi-parabolas given in the preceding table to find the altitudes of each of the parabolas described with the same initial speed.

Let *bc* denote the given amplitude; and let *ob*, the sum of the altitude and sublimity, be the measure of the initial speed which is understood to remain constant. Next we must find and determine the altitude, which we shall accomplish by so dividing *ob* that the rectangle contained by its parts shall be equal to the square of half the amplitude, *bc*. Let *f* denote this point of division and *d* and *i* be the middle points of *ob* and *bc* respectively. The square of *ib* is equal to the rectangle *bf.fo*; but the square of *do* is equal to the sum of the rectangle *bf.fo* and the {286} square of *fd*. If, therefore, from the square of *do* we subtract the square of *be* which is equal to the rectangle *bf.fo*, there will remain the square of *fd*. The altitude in question, *bf*, is now obtained by adding to this length, *fd*, the line *bd*. The process is then as follows: From the square of half of *bo* which is known, subtract the square of *be* which is also known; take the square root of the remainder and add to it the known length *db*; then you have the required altitude, *bf*.

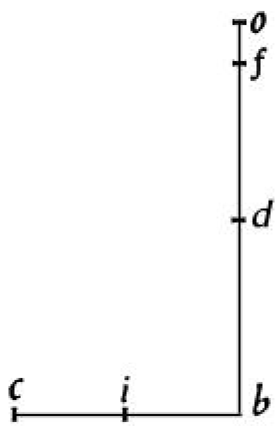

Fig. 125

Example. To find the altitude of a semi-parabola described with an angle of elevation of 55°. From the preceding table the amplitude is seen to be 9396, of which the half is 4698, and the square 22071204. When this is subtracted from the square of the half of *bo*, which is always 25000000, the remainder is 2928796, of which the square root [306] is approximately 1710. Adding this to the half of *bo*, namely 5000, we have 6710 for the altitude of *bf*.

It will be worth while to add a third table giving the altitudes and sublimities for parabolas in which the amplitude is a constant.

SAGR. I shall be very glad to see this; for from it I shall learn the difference of speed and force [*degl'impeti e delle forze*] required to fire projectiles over the same range with what we call mortar shots. This difference will, I believe, vary greatly with the elevation so that it, for example, one wished to employ an elevation of 3° or 4°, or 87° or 88° and yet give the ball the same range which it had with an elevation of 45° (where we have shown the initial speed to be a minimum) the excess of force required will, I think, be very great.

SALV. You are quite right, sir; and you will find that in order to perform this operation completely, at all angles of elevation, you will have to make great strides toward an infinite speed. We pass now to the consideration of the table.

[307]

Table giving the altitudes and sublimities of parabolas of constant amplitude, namely 10000, computed for each degree of elevation

Angle of Elevation°	Altitude	Sublimity	Angle of Elevation°	Altitude	Sublimity
1	87	286533	46	5177	4828
2	175	142450	47	5363	4662
3	262	95802	48	5553	4502
4	349	71531	49	5752	4345
5	437	57142	50	5959	4196
6	525	47573	51	6174	4048
7	614	40716	52	6399	3906
8	702	35587	53	6635	3765
9	792	31565	54	6882	3632
10	881	28367	55	7141	3500
11	972	25720	56	7413	3372
12	1063	23518	57	7699	3247
13	1154	21701	58	8002	3123
14	1246	20056	59	8332	3004
15	1339	18663	60	8660	2887
16	1434	17405	61	9020	2771
17	1529	16355	62	9403	2658
18	1624	15389	63	9813	2547
19	1722	14522	64	10251	2438
20	1820	13736	65	10722	2331
21	1919	13024	66	11230	2226
22	2020	12376	67	11779	2122
23	2123	11778	68	12375	2020
24	2226	11230	69	13025	1919
25	2332	10722	70	13237	1819
26	2439	10253	71	14521	1721
27	2547	9814	72	15388	1624
28	2658	9404	73	16354	1528

Table giving the altitudes and sublimities of parabolas of constant amplitude, namely 10000, computed for each degree of elevation {cont.}

Angle of Elevation°	Altitude	Sublimity	Angle of Elevation°	Altitude	Sublimity
29	2772	9020	74	17437	1433
30	2887	8659	75	18660	1339
31	3008	8336	76	20054	1246
32	3124	8001	77	21657	1154
33	3247	7699	78	23523	1062
34	3373	7413	79	25723	972
35	3501	7141	80	28356	881
36	3633	6882	81	31569	792
37	3768	6635	82	35577	702
38	3906	6395	83	40222	613
39	4049	6174	84	47572	525
40	4196	5959	85	57150	437
41	4346	5752	86	71503	349
42	4502	5553	87	95405	262
43	4662	5362	88	143181	174
44	4828	5177	89	286449	87
45	5000	5000	90	*Infinita*	

[308] {288}

PROPOSITION XIV

To find for each degree of elevation the altitudes and sublimities of parabolas of constant amplitude.

The problem is easily solved. For if we assume a constant amplitude of 10000, then half the tangent at any angle of elevation will be the altitude. Thus, to illustrate, a parabola having an angle of elevation of 30° and an amplitude of 10000, will have an altitude of 2887, which is approximately one-half the tangent. And now the altitude having been found, the sublimity is derived as follows. Since it has been proved that half the amplitude of a semi-parabola is the mean proportional between the altitude and sublimity, and since the altitude has already been found, and since the semi-amplitude is a constant, namely 5000, it follows that if we divide the square of the semi-amplitude by the altitude we shall obtain the sublimity sought. Thus in our example the altitude was found to be 2887; the square of 5000 is 25,000,000, which divided by 2887 gives the approximate value of the sublimity, namely 8659.

SALV. Here we see, first of all, how very true is the statement made above, that, for different angles of elevation, the greater the deviation from the mean, whether above or below, the greater the initial speed [*impeto e violenza*] required to carry the projectile over the same range. For since the speed is the resultant of two motions, namely, one horizontal and uniform, the other vertical and naturally accelerated; and since the sum of the altitude and sublimity represents this speed, it is seen from the preceding table that this sum is a {289} minimum for an elevation of 45° where the altitude and sublimity are equal, namely, each 5000; and their sum 10000. But if we choose a greater elevation, say 50°, we shall find the altitude 5959, and the sublimity 4196, giving a sum of 10155; in like manner we shall find that this is precisely the value of the speed at 40° elevation, both angles deviating equally from the mean.

Secondly it is to be noted that, while equal speeds are required for each of two elevations that are equidistant from the mean, there is this curious alternation, namely, that the altitude and sublimity at the greater elevation correspond inversely to the sublimity and altitude at the lower elevation. Thus in the [309] preceding example an elevation of 50° gives an altitude of 5959 and a sublimity of 4196; while an elevation of 40° corresponds to an altitude of 4196 and a sublimity of 5959. And this holds true in general; but it is to be remembered that, in order to escape tedious calculations, no account has been taken of fractions which are of little moment in comparison with such large numbers.

SAGR. I note also in regard to the two components of the initial speed [*impeto*] that the higher the shot the less is the horizontal and the greater the vertical component; on the other hand, at lower elevations where the shot reaches only a small height the horizontal component of the initial speed must be great. In the case of a projectile fired at an elevation of 90°, I quite understand that all the force [*forza*] in the world would not be sufficient to make it deviate a single finger's breadth from the perpendicular and that it would necessarily fall back into its initial position; but in the case of zero elevation, when the shot is fired horizontally, I am not so certain that some force, less than infinite, would not carry the projectile some distance; thus not even a cannon can fire a shot in a perfectly horizontal direction, or as we say, point blank, that is, with no elevation at all. Here I admit there is some room for doubt. The fact I do not deny outright, because of another phenomenon apparently no less remarkable, but yet one for which I have conclusive evidence. This phenomenon is the impossibility of stretching {290} a rope in such a way that it shall be at once straight and parallel to the horizon; the fact is that the cord always sags and bends and that no force is sufficient to stretch it perfectly straight.

SALV. In this case of the rope than, Sagredo, you cease to wonder at the phenomenon because you have its demonstration; but if we consider it with more care we may possibly discover some correspondence between the case of the gun and that of the string. The curvature of the path of the shot fired horizontally appears to result from two forces, one (that of the weapon) drives it horizontally and the other (its own weight) draws it vertically downward. So in stretching the rope you have the force which pulls it horizontally and its own weight which acts downwards. The circumstances in these two cases are, therefore, very similar. If then you attribute to the weight of the rope a power and [310] energy [*possanza ed energia*] sufficient to oppose and overcome any stretching force, no matter how great, why deny this power to the bullet?

Besides I must tell you something which will both surprise and please you, namely, that a cord stretched more or less tightly assumes a curve which closely approximates the parabola. This similarity is clearly seen if you draw a parabolic curve on a vertical plane and then invert it so that the apex will lie at the bottom and the base remain horizontal; for, on hanging a chain below the base, you will observe that, on slackening the chain more or less, it bends and fits itself to the parabola; and the coincidence is more exact in proportion as the parabola is drawn with less curvature or, so to speak, more stretched; so that using parabolas described with elevations less than 45° the chain fits its parabola almost perfectly.

SAGR. Then with a fine chain one would be able to quickly draw many parabolic lines upon a plane surface.

SALV. Certainly and with no small advantage as I shall show you later.

SIMP. But before going further, I am anxious to be convinced at least of that proposition of which you say that there is a {291} rigid demonstration; I refer to the statement that it is impossible by any force whatever to stretch a cord so that it will lie perfectly straight and horizontal.

SAGR. I will see if I can recall the demonstration; but in order to understand it, Simplicio, it will be necessary for you to take for granted concerning machines what is evident not alone from experiment but also from theoretical considerations, namely, that the velocity of a moving body [*velocità del movente*], even when its force [*forza*] is small, can overcome a very great resistance exerted by a slowly moving body, whenever the velocity of the moving body bears to that of the resisting body a greater ratio than the resistance [*resistenza*] of the resisting body to the force [*forza*] of the moving body.

SIMP. This I know very well for it has been demonstrated by Aristotle in his *Questions in Mechanics*; it is also clearly seen in the lever and the steelyard where a counterpoise weighing not more than 4 pounds will lift a weight of 400 provided that the distance of the counterpoise from the axis about which the steelyard rotates be more than one hundred times as great as the distance between this axis and the point of support for [311] the large weight. This is true because the counterpoise in its descent traverses a space more than one hundred times as great as that moved over by the large weight in the same time; in other words the small counterpoise moves with a velocity which is more than one hundred times as great as that of the large weight.

SAGR. You are quite right; you do not hesitate to admit that however small the force [*forza*] of the moving body it will overcome any resistance, however great, provided it gains more in velocity than it loses in force and weight [*vigore e gravità*]. Now let us return to the case of the cord. In the accompanying figure *ab* represents a line passing through two fixed points *a* and *b*; at the extremities of this line hang, as you see, two large weights *c* and *d*, which stretch it with great force and keep it truly straight, seeing that it is merely a line without weight. Now I wish to remark that if from the middle point of this line, {292} which we may call *e*, you suspend any small weight, say *h*, the line *ab* will yield toward the point *f* and on account of its elongation will compel the two heavy weights *c* and *d* to rise. This I shall demonstrate as follows: with the points *a* and *b* as centers describe the two quadrants, *eig* and *elm*; now since the two semi-diameters *ai* and *bl* are equal to *ae* and *eb*, the remainders *fi* and *fl* are the excesses of the lines *af* and *fb* over *ae* and *eb*; they therefore determine the rise of the weights as *c* and *d*, assuming of course that the weight *h* has taken the position *f*. But the weight *h* [312]will take the

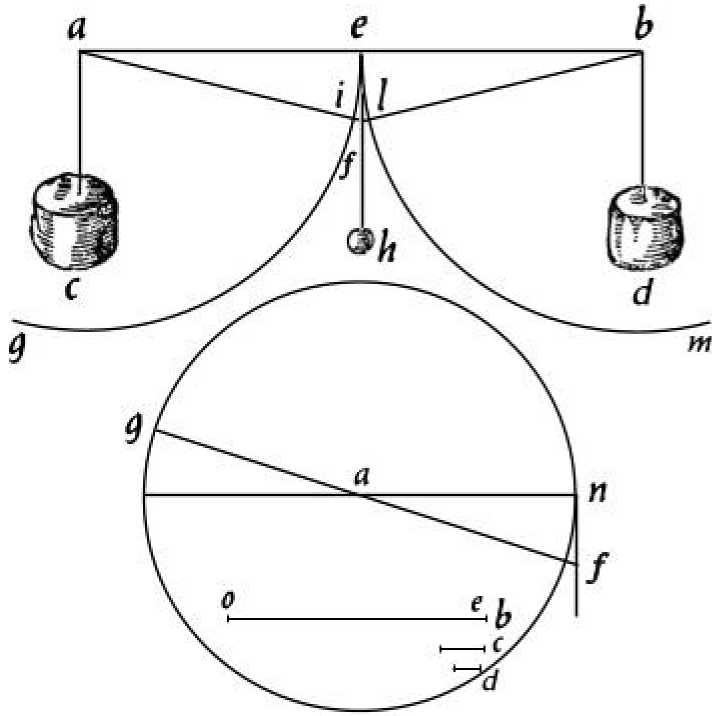

Fig. 126

position *f*, whenever the line *ef* which represents the descent of *h* bears to the line *fi*—that is, to the rise of the weights *c* and *d*—a ratio which is greater than the ratio of the weight of the two large bodies to that of the body *h*. Even when the weights of *c* and *d* are very great and that of *h* very small this will happen; for the excess of the weights *c* and *d* over the weight of *h* can never be so great but that the excess of the tangent *ef* over the segment *fi* may be proportionally greater. This may {293} be proved as follows: Draw a circle of diameter *gai*; draw the line *bo* such that the ratio of its length to another length *c*, $c > d$, is the same as the ratio of the weights *c* and *d* to the weight *h*. Since $c > d$, the ratio of *bo* to *d* is greater than that of *bo* to *c*. Take *be* a third proportional to *ob* and *d*; prolong the diameter *gi* to a point *f* such that $gi:if = oe:eb$; and from the point *f* draw the tangent *fn*; then since we already have $oe:eb = gi:if$, we shall obtain, by compounding ratios, $ob:eb = gf:if$. But *d* is a mean proportional between *ob* and *be*; while *nf* is a mean proportional between *gf* and *fi*. Hence *nf* bears to *fi* the same ratio as that of *cb* to *d*, which is greater than that of the weights *c* and *d* to the weight *h*. Since then the descent, or velocity, of the weight *h* bears to the rise, or velocity, of the weights *c* and *d* a greater ratio than the weight of the bodies *c* and *d* bears to the weight of *h*, it is clear that the weight *h* will descend and the line *ab* will cease to be straight and horizontal.

And now this which happens in the case of a weightless cord *ab* when any small weight *h* is attached at the point *e*, happens also when the cord is made of ponderable matter but without any attached weight; because in this case the material of which the cord is composed functions as a suspended weight.

SIMP. I am fully satisfied. So now Salviati can explain, as he promised, the advantage of such a chain and, afterwards, present the speculations of our Academician on the subject of impulsive forces [*forza della percossa*].

SALV. Let the preceding discussions suffice for to-day; the hour is already late and the time remaining will not permit us to clear up the subjects proposed; we may therefore postpone our meeting until another and more opportune occasion.

SAGR. I concur in your opinion, because after various conversations with intimate friends of our Academician I have concluded that this question of impulsive forces is very obscure, and I think that, up to the present, none of those who have treated [313] this subject have been able to clear up its dark corners which lie almost beyond the reach of human imagination; among the various views which I have heard expressed one, strangely fantastic, {294} remains in my memory, namely, that impulsive forces are indeterminate, if not infinite. Let us, therefore, await the convenience of Salviati. Meanwhile tell me what is this which follows the discussion of projectiles.

SALV. These are some theorems pertaining to the centers of gravity of solids, discovered by our Academician in his youth, and undertaken by him because he considered the treatment of Federigo Comandino to be somewhat incomplete. The propositions which you have before you would, he thought, meet the deficiencies of Comandino's book. The investigation was undertaken at the instance of the Illustrious Marquis Guid'Ubaldo Dal Monte, a very distinguished mathematician of his day, as is evidenced by his various publications. To this gentleman our Academician gave a copy of this work, hoping to extend the investigation to other solids not treated by Comandino. But a little later there chanced to fall into his hands the book of the great geometrician, Luca Valerio, where he found the subject treated so completely that he left off his own investigations, although the methods which he employed were quite different from those of Valerio.

SAGR. Please be good enough to leave this volume with me until our next meeting so that I may be able to read and study these proposition in the order in which they are written.

SALV. It is a pleasure to comply with your request and I only hope that the propositions will be of deep interest to you.

END OF THE FOURTH DAY.

Galileo Galilei, *Dialogues Concerning Two New Sciences*, translated by Henry Crew & Alfonso de Salvio with an introduction by Antonio Favaro, Dover Publications, Inc., New York, 1954: 244–294. (Fourth Day). Originally published in 1904 by the MacMillan company.

APPENDIX

Containing some theorems, their proofs, dealing with centers of gravity of solid bodies, written by the same Author at an earlier date.*

* Following the example of the National Edition, this Appendix which covers 18 pages of the Leyden Edition of 1638 is here omitted [in the Crew & De Salvio edition (*Transc*)] as being of minor interest.

[FINIS]

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INTRODUCTION FIRST DAY SECOND DAY THIRD DAY FOURTH DAY

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THE ADDED (OR "FIFTH" DAY) BY STILLMAN DRAKE (1974)
EARLIER APPENDIX INCLUDED BY STILLMAN DRAKE (1974)

Galileo Galilei, *Discourses and Mathematical Demonstrations Concerning Two New Sciences Pertaining to Mechanics and Local Motions.* Translated by Stillman Drake, University of Wisconsin Press, Madison, 1974: 281–303. (ADDED DAY)