Galileo Galilei

Lincean Academician

Chief Philosopher and Mathematician to the
Most Serene Grand Duke of Tuscany

Discourses
&
Mathematical Demonstrations
Concerning

Two New Sciences

Pertaining to
Mechanics & Local Motions

With an Appendix
On Centers of Gravity of Solids

Leyden
At the Elzevirs, 1638

*   *   *

To which is added a further dialogue
On the Force of Percussion

Galileo Galilei

Two New Sciences

Including Centers of Gravity & Force of Percussion

Translated, with
Introduction and Notes, by

Stillman Drake

[THE APPENDIX]

TRANSCRIBER'S NOTES (Added)

The "Appendix" included in Stillman Drake's 1974 translation of Galileo's *Two New Sciences* (1638) is presented here with a number of minor cosmetic changes intended to render the work more readable in Portable Document Form (PDF). To this end line spaces have been introduced to emphasize the various lemmas, propositions and postulates discussed in the text. For additional clarity the original pagination has been omitted entirely, and except when occurring as natural breaks between sections modern page numbers denoted here by {NN} have been included within the text. Rather than retaining the large margins and accompanying small marginal figures of the 1974 publication enlarged figures have been incorporated within the text to match the format adopted for the "Added Day" and previous versions of the *Two New Sciences* from the present source:

Galileo Galilei, *Dialogues Concerning Two New Sciences*, translated by Henry Crew & Alfonso de Salvio, with an introduction by Antonio Favaro, Dover Publications, Inc., New York, 1954.

INTRODUCTION FIRST DAY SECOND DAY THIRD DAY FOURTH DAY

Appendix

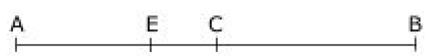
In which are contained theorems and related demonstrations concerning the center of gravity of solids, written earlier by the Author[1]

POSTULATE

We assume that, of equal weights similarly arranged on different balances, if the center of gravity of one composite [of weights] divides its balance in a certain ratio, then the center of gravity of the other composite also divides its balance in the same ratio.

LEMMA

Let line AB *be bisected at* C, *and the half* AC *be divided at* E *so that the ratio of* BE *to* EA *is that of* AE *to* EC. *I say that* BE *is double* EA.



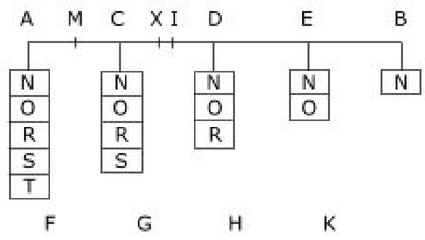
Indeed, since *EA* is to *EC* as *BE* is to *EA*, we shall have, by composition and permutation [of ratios], *AE* to *EC* as *BA* is to *AC;* but as *AE* is to *EC* (that is, as *BA* is to *AC)*, *BE* is to *EA*, whence *BE* is double *EA*.

These things granted, it is to be demonstrated [that]

[PROPOSITION 1]

If any number of magnitudes equally exceed one another the {262} *excesses being equal to the least of them, and they are so arranged on a balance as to hang at equal distances, the center of gravity of all these divides the balance so that the part on the side of the smaller* [*magnitudes*] *is double the other part.*

1. These theorems date, in part at least, from the period 1585-87. The last proposition and its lemma appear to have been written first, having been submitted by Galileo with an application for a position at the University of Bologna in 1587. Early in the next year he corresponded with Christopher Clavius and Guidobaldo del Monte about the first proposition. The others may have been done in response to encouragement from the latter and from Michael Coignet (1544-1623) at that time. A plan to publish this work in 1613 was postponed, cf. note 37 to Second Day. In the original printing the lemmas, theorems, and corollaries were not numbered, and they were not always clearly distinguished typographically both have been done here for ease of reference.



Thus, on balance *AB*, let hang at equal distances any number of magnitudes *F,,G, H, K, N,* such as described above, of which the least is *N*, let the points of suspension be *A, C, D, E, B,* and let *A* be the center of gravity of all the magnitudes thus arranged. It is to be shown that the part of the balance *BX,* on the side of the lesser magnitudes, is double *XA,* the other part.

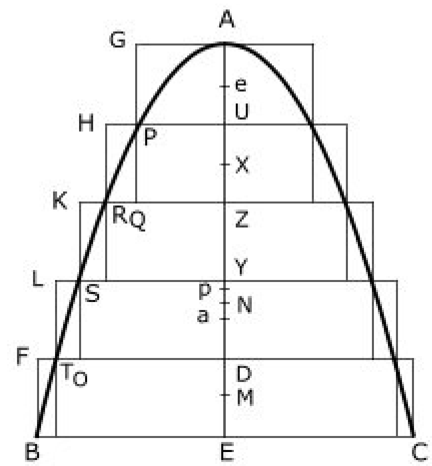
Bisect the balance at point *D*, which lies either at some point of suspension, or necessarily falls midway between two suspension points. The remaining distances between suspension [points], *A* and [*C, C* and] *D*, are to be bisected at points *M* and *I*, and all the magnitudes are to be divided into parts equal to *N*. Then the number of parts of *F* will be equal to the number of magnitudes that hang from the balance, while the parts of *G* will be one fewer, and so on for the rest. Thus the parts of *F* are *N, O, R, S, T;* those of *G* [are] *N, O, R, S,* those of *H* [are] *N, O, R,* and finally the parts of *K* are *N* and *O*. All the parts marked *N* are then equal to [those in] *F;* all the parts marked *O* will be equal to *G,* those marked *R* will be equal to *H,* those marked *S* will be equal to *K;* and finally the magnitude *T* is equal to *N*.

Since all the magnitudes marked *N* are equal to one another, their point of balance will be at *D,* which bisects the balance *AB.* For the same reason, the point of balance for all the magnitudes marked *O* is at *I* ; of those marked *R,* it is at *C*; those marked *S* have their point of balance at *M*, while finally *T* is hung at *A*. Thus along the balance *AD,* [considered as separated from *DB*], there are hung, at the equal distances *D I, C M, A,* magnitudes that equally exceed one another and whose excess is equal to the least thereof. But [of these] the greatest [magnitude], composed of all the *N*'s, hangs [as if] from *D*, while the least (that is, *T)* hangs from *A,* and the others are all arranged in order.

And again, there is the other balance *AB* on which corresponding magnitudes are arranged in the same order [though reversed], equal in number and sizes to the foregoing. Wherefore we see the balances *AB* and *AD* divided in the same ratio by the centers [of gravity] of all the magnitudes {263} compounded. But the center of gravity of the said magnitudes [so arranged] is *X;* [2] therefore *X* divides the balances *BA* and *AD* in the same ratio, in such a way that as *BX* is to *XA,* so *XA* is to *XD.* Therefore *BX* is double *XA,* by the above lemma. Q. E. D.

[PROPOSITION 2]

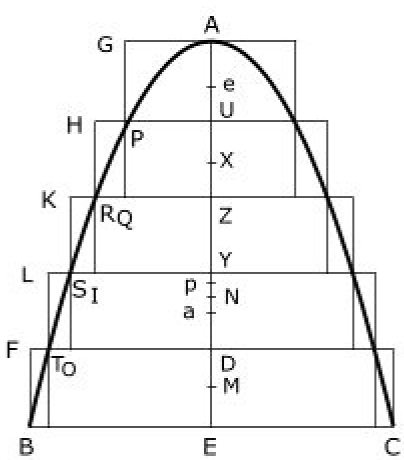
If to a parabolic conoid one figure is inscribed and another is circumscribed,[*both*] *of cylinders having equal height, and the axis of the conoid is divided in such a way that the part toward the apex is double the part toward the base, the center of gravity of the inscribed figure will be closer to the base of the section than [will] the said division point, while the center of gravity of the circumscribed figure will be farther than that same point from the base of the conoid and the distance from that point of each of the two centers will be equal to the line that is one-sixth the height of one of the cylinders of which the figures are constructed.*



Let there be a parabolic conoid and the said figures, one inscribed and the other circumscribed, let the axis of the conoid be *AE,* divided at *N* so that *AN* is double *NE.* It is to be shown that the center of gravity of the inscribed figure lies in line *NE,* while that of the circumscribed figure lies in *AN*.

Let the figures thus arranged be cut by a plane through the axis, and let the parabola *BAC* be cut, the [inter]-section of the cutting plane with the base of the conoid being line *BC;* the sections of the cylinders are rectangular figures, as appears in the diagram.

The first inscribed cylinder, of which the axis is *DE,* has to the cylinder of which the axis is *DY* the same ratio that the square [on] *ID* has to the square [on] *SY,* which is [in turn] as *DA* is to *AY.* [3] The cylinder of which the axis is *DY* is, moreover, to the cylinder *YZ* as the square on *SY* is to the square on *RZ,* which is as *YA* to *AZ,* and for the same reason the cylinder of which the axis is *ZY*, to that of which the axis is *ZU*, is as *ZA* is to *AU*. Thus the said cylinders are to one another {264} as the lines *DA, AY , ZA, AU;* but these [lines] equally exceed one another, and the excess is equal to the least of them, hence *AZ* is the double of *AU, AY is* its triple, and *DA* its quadruple. Therefore the said cylinders are magnitudes equally exceeding one another, whose excess is equal to the least of them. Moreover, line *XM* is that along which these are hung at equal distances (indeed, each cylinder has its center of gravity at the midpoint of its own axis), whence, by the things previously demonstrated, the center of gravity of the magnitude composed of all [these] magnitudes divides the line *XM* so that the part toward *X* is double the remainder. Let it be divided thus, and let *Xa* be double *aM,* then point *a* is the center of gravity of the inscribed figure.



Let *AU* be bisected at point *e, eX* will be double *ME;* but *Xa* is double *aM,* whence *eE* will be triple *Ea.* Further, *AE* is triple *EN*, thus it is clear that *EN* is greater than *Ea,* and for that reason point *a,* which is the center of the inscribed figure, more nearly approaches to the base of the conoid than [does] *N*. And since as *AE* is to *EN*, so the removed part *eE* is to the removed part *Ea,* the remainder will be to the remainder (that is, *Ae* [will be] to *Na)* as *AE* is to *EN*. Therefore *aN* is one-third

2. Both Clavius and Guidobaldo (note 1, above) believed this assumption to beg the question. The latter was satisfied by Galileo's explanation, sent to him in 1588 with a redrawn diagram showing all the weights as touching horizontally; cf. p. 198.

3. It was a well known property of the parabola that the squares on the abscissae are in the ratio of the ordinates, but cf. note 4, below.

of Ae, and one-sixth of AU.

Further, the cylinders of the circumscribed figure will be shown in the same way to exceed one another equally, the excess being equal to the least of them, and to have their centers of gravity equidistant along line eM. Hence if eM is divided at p so that ep is double the remainder pM, then p will be the center of gravity of the whole circumscribed magnitude, and since ep is double pM, and Ae is less than double EM (for these are equal), all AE is less than triple Ep, whence Ep will be greater than EN. And since eM is triple Mp, and ME plus double eA is likewise triple ME, all AE plus Ae will be triple Ep. But AE is triple EN, so the remainder Ae will be triple the remainder pN. Therefore Np is one-sixth of AU. But these were the things to be proved. And from this it is manifest that:

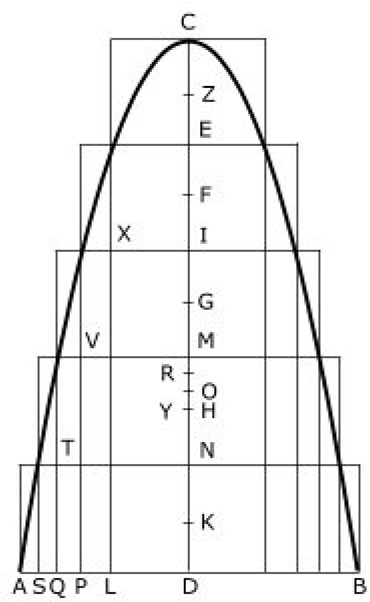
[COROLLARY]

To a parabolic conoid, one figure may be inscribed and another circumscribed so that their centers of gravity may be made less distant from N *than any assigned length.*

In fact, if a line is taken six times the assigned length, {265} and the axes of the cylinders composing those figures are made less than the said line, then the distances between the [respective] centers of gravity of these [two] figures and the point N will [both] be less than the assigned line.

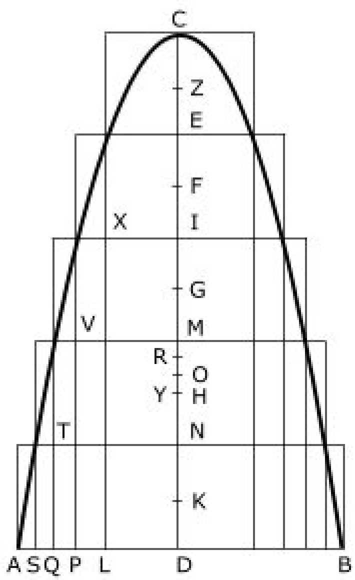
The same [proposition], otherwise [demonstrated]:

Let CD be the axis of a conoid, so divided at O that CO is double OD. It must be shown that the center of gravity of the inscribed figure lies in OD, while the center of the circumscribed [figure] lies in CO.



As above, the figures are intersected by a plane through the axes and through C. Now, cylinders SN, TM, VI, and XE are to one another as the squares on lines SD, TN, VM, and XI; and these are to one another as are lines NC, CM, CI and CE, which moreover exceed one another equally, and this excess is equal to the least [of them], which is CE; and cylinder TM equals cylinder QN, while cylinder VI equals cylinder PN, and cylinder XE equals cylinder LN; therefore cylinders SN, QN, PN and LN exceed one another equally and the excess is equal to the least of these, that is, to cylinder LN. But the excess of cylinder SN over cylinder QN is a ring of height QT (or ND) and of breadth SQ, the excess of cylinder QN over cylinder PN is a ring of breadth QP; and finally the excess of cylinder PN over cylinder LN is a ring of breadth PL. Hence the said rings SQ, QP, PL are equal [in volume] to one another and to cylinder LN. Ring ST is therefore equal to cylinder XE; ring QV, double ring *ST,* is equal to cylinder *VI,* which is likewise double the cylinder *XE;* and for the same reason, ring *PX* will be equal to cylinder *TM,* and cylinder *LE* [equal] to cylinder *SN*.

Therefore along the balance *KF,* which joins the midpoints of lines *El* and *DN* and is cut into equal parts by points *H* and *G*, there are magnitudes (that is, cylinders *SN*, *TM VI*, and *XE)* of which the centers of gravity are respectively *K*, *H*, *G* and *F*. Further, we have another balance, *MK,* which is one-half *FK,* and which is divided into as many equal parts by as many points, that is, [lines] *MH, HN* and *NK;* and on this there are other magnitudes equal in number and size to those found on the balance *FK*, having their centers of gravity at points *M*, *H*, *N*, *K* and being arranged in the same order. In fact, cylinder *LE* has its center of gravity at M and is equal to cylinder *SN*, which has its center of gravity at *K*, ring *PX* has its center of gravity at *H* and is equal to the cylinder *TM,* of which the center of gravity is {266} at *H*, ring *QV*, having its center of gravity at *N*, is equal to cylinder *VI*, of which the center is *G*, finally, ring *ST*, having its center of gravity at *K*, equals cylinder *XE* of which the center is at *F*. Therefore the center of gravity of [each of] the said magnitudes divides the [respective] balance in the same ratio. But their center [of gravity] is unique, and is therefore at some point common to both balances, let this be *Y*. Hence *FY* will be to *YK* as *KY* is to *YM*, therefore *FY* is double *YK;* and, *CE* being bisected at *Z*, *ZF* will be double *KD,* and consequently *ZD* will be triple *DY*. But *CD* is triple *DO*, therefore line *DO* is greater than *DY*, and hence the center of gravity *Y* of the inscribed figure is closer to the base than is the point *O*. And since as *CD* is to *DO*, so the removed part *ZD* is to the removed part *DY*, then the remainder *CZ* will also be to the remainder *YO*, as *CD* is to *DO*, that is, *YO* will be one-third of *CZ*, or one-sixth *of CE*.



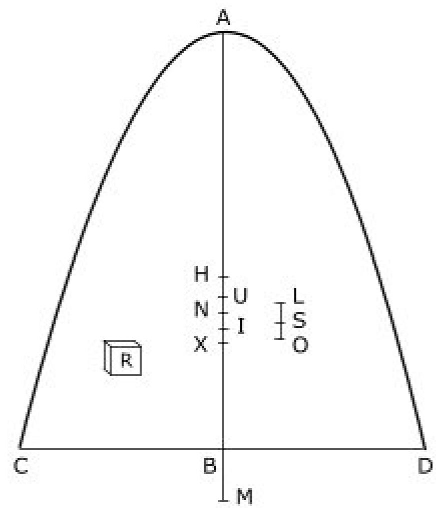
By the same procedure we may show, on the other hand, that the cylinders of the circumscribed figure exceed one another equally, that their excesses are equal to the minimum cylinder, and that their centers of gravity are situated at equal distances along balance *KZ,* and likewise the rings equal to the cylinders are disposed in a like manner along the balance *KG,* which is one-half of balance *KZ,* and that hence the center of gravity *R* of the circumscribed figure divides the balance so that *ZR* is to *RK* as *KR* is to *RG*. Therefore *ZR* will be double *RK;* but *CZ* will be equal to line *KD,* and not its double, hence all *CD* will be less than triple *DR,* and so line *DR* is greater than *DO,* or the center of gravity of the circumscribed figure is farther from base than is the point *O*. And since *ZK* is triple *KR*, and *KD* plus double *ZC* is triple *KD,* all *CD* plus *CZ* will be triple *DR*. But *CD* is triple *DO,* hence the remainder *CZ* will be triple the other remainder *RO,* that is, *OR* is one-sixth of *EC.* Which was the proposition.

These things first demonstrated, it will be proved that:

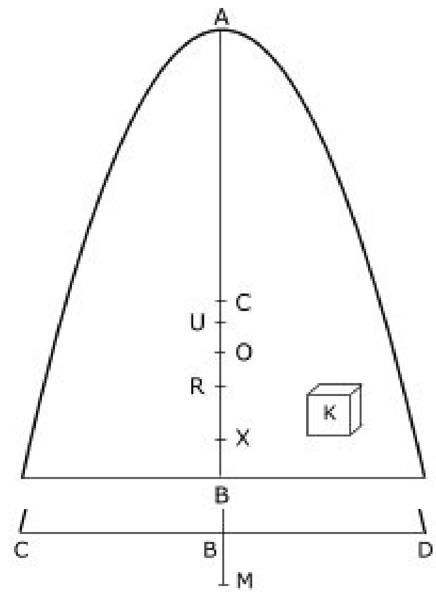
[PROPOSITION 3]

The center of gravity of a parabolic conoid divides its axis so that the part toward the vertex is double the part toward the base.

The parabolic conoidal [figure] whose axis is *AB* is divided at *N* so that *AN* is double *NB*. It is to be showed that the center {267} of gravity of the conoid is point *N*. If, indeed, it is not *N*, it is below this [point] or above it. First let it be below, at *X*, and draw *LO* equal to *NX*, and let *LO* be divided anywhere at *S*; and whatever ratio *BX* plus *OS* has to *OS*, let the [volume of the] conoid have to the solid R.



Inscribe in the conoid a figure made up of cylinders of equal height in such a way that between its center of gravity and the point *N*, [a distance] less than *LS* shall be intercepted, and let the excess by which the conoid exceeds it be less than the solid *R*. It is manifest that this can be done. Thus let the inscribed [figure] be that of which the center of gravity is *I* ; now *IX* will be greater than *SO,* and since as *XB* plus *SO* is to *SO,* so the conoidal [figure] is to *R*, and further, *R* is greater than the excess by which the conoid exceeds it, the ratio of the conoid to the said excess will be greater than *BX* plus *OS* to *SO*, and by division, the inscribed figure will have a greater ratio to the said excess than *BX* has to *SO*. But *BX* has to *XI* a smaller ratio than to *SO,* therefore the inscribed figure will have to the remaining parts a much greater ratio than *BX* [has] to *XI.* Therefore the ratio of the inscribed figure to the remaining parts will be that of some other line to *XI*, which [line] must be greater than *BX*. Let it be *MX*. Thus we have *X*, the center of gravity of the conoid, but the center of gravity of the inscribed figure is *I*. Therefore the center of gravity of the remaining portions, by which the conoid exceeds the inscribed figure, will be in the line *XM,* and at that point wherein it terminates so that the ratio of the inscribed figure to the excess by which the conoid surpasses it is the same as [the ratio of] this [line] to *XI.* But it has been shown that this ratio is that of *MX* to *XI;* therefore *M* will be the center of gravity of the portions by which the conoid exceeds the inscribed figure. But that certainly cannot be, for if a plane is drawn through *M,* parallel to the base of the conoid, all the said [excessive] parts will lie on the same side of it and will not be divided by it. Therefore the center of gravity of the conoid is not below point *N*.



But neither is it above. Indeed, if this is possible, let it be [at] *H;* and as above, draw *LO* equal to *HN* and divide this anywhere at *S;* and whatever ratio *BN* plus *SO* has to *SL,* let the conoid have to *R.* Circumscribe about the conoid a figure [composed] of cylinders, as before, exceeding the conoid by a quantity less than the solid *R,* and let the line between the center of gravity of the circumscribed figure and point *N* be less than *SO.* The remainder *UH* will be {268} greater than *LS,* and since as *BN* plus *OS* is to SL, so the conoid is to *R* (*R* being greater than the excess by which the circumscribed figure exceeds the conoid), then *BN* plus *SO* has a smaller ratio to *SL* than the conoid has to the said excess. But *BU* is less than *BN* plus *SO,* while *HU* is greater than *SL,* whence the conoid has a much greater ratio to the said portions [of excess] than *BU* has to *UH* Therefore whatever ratio the conoid has to the said portions, some line greater than *BU* has to *UH.* Let this be *MU,* and since the center of gravity of the circumscribed figure is *U,* while the center of the conoid is *H,* and as the conoid is to the remaining portions, so *MU* is to *UH,* then *M* will be the center of gravity of the remaining portions, which likewise is impossible. Therefore the center of gravity of the conoid is not above the point *N* But it was demonstrated not to be below it, therefore it necessarily lies at TV And by the same reasoning this may be proved for a conoid cut by a plane that is not at right angles to its axis.

The same is shown in another way, as is clear from the following

[PROPOSITION 4]

The center of gravity of a parabolic conoid falls between the center of the circumscribed figure [*of cylinders*] *and the center of the* [*similar*] *inscribed figure.*

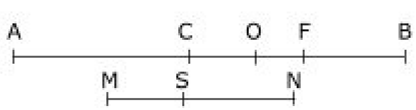
Let there be a conoid with axis *AB;* the center [of gravity] of the circumscribed figure is *C,* while that of the inscribed figure is *O.* I say that the center [of gravity] of the conoid lies between points C and *O.* Indeed, if it does not, it lies either above, or below, or at one of these [points]. Let it be below, as for example at *R,* then since *R* is the center of gravity of the whole conoid and *O* is the center of gravity of the inscribed figure, the center of gravity of all the other portions by which the inscribed figure is exceeded by the conoid will lie on the extension of line *OR* beyond *R,* and precisely at that point which terminates it in such a way that whatever ratio the said portions have to the inscribed [figure], that is also the ratio of line *OR* to the line intercepted between *R* and that point. Let this ratio be that of *OR* to *RX,* then *X* will either fall outside the conoid, or inside it, or in its base. That it should fall outside, or in the base, is clearly {269} absurd. Falling inside, since *XR* is to *RO* as the inscribed figure is to the excess by which this is surpassed by the conoid, then we assume that whatever the ratio of *BR* to *RO,* such also is that of the inscribed figure to the solid *K,* which must necessarily be less than that excess.

Next, inscribe another figure which shall be exceeded by the conoid by an excess less than A"; its center of gravity will lie between *O* and *C*. Let this be *U;* since the first figure is to *K* as *BR* is to *RO*, and since on the other hand the second figure, of which the center is *U,* is greater than the first, and is exceeded by the conoid with an excess less than *K,* we shall have that whatever the ratio of the second figure to the excess by which it is surpassed by the conoid, such also is the ratio of some line greater than *BR* to line *RU* But the center of gravity of the conoid is *R,* while that of the inscribed figure is £7; therefore the center of gravity of the remaining portions will lie outside the conoid, below *B,* which is impossible.

By the same procedure it will be shown that the center of gravity of this same conoid does not lie on line *CA.* Then, that it is neither of the points C or O is manifest. In fact if we suppose this, and describe other figures such that the inscribed is greater than the figure whose center [of gravity] is *O,* and that which is circumscribed is less than the figure whose center is *C,* the center of gravity of the conoid will fall outside the centers of gravity of these figures, which is impossible, as we have just concluded. It follows, then, that it lies between the center of the circumscribed figure and that of the inscribed figure. Being thus, it must necessarily lie in that point that divides the axis in such a way that the part toward the vertex is double the remainder, since indeed figures can be inscribed and circumscribed such that the lines lying between their centers of gravity and the said point may be less than any given line. Thus anyone who declared the contrary [of the above] would be led to the absurdity that the center [of gravity] of the conoid would not lie between the centers of gravity of the inscribed and circumscribed figures.

[LEMMA]

If there are three lines in [*continued*] *proportion, and the ratio of the least to the excess by which the greatest exceeds the least is the same as that of some given line to two-thirds of the excess by which the greatest exceeds the middle* [*line*] {270} *and again if the ratio of the greatest plus double the middle* [*line*] *to triple the greatest plus triple that middle is the same as the ratio of some* [*other*] *given line to the excess of the greatest over the smallest, then the sum of those two given lines is one-third of the greatest of the three proportional lines.*



Let there be three lines, *AB, BC BF,* in [continued] proportion, and let the ratio of *BF* to *AF* be that of *MS* to two-thirds of *CA,* also let the ratio of *AB* plus *2BC* to

3AB plus *3BC* be that of another [line] *SN* to *AC*. It is to be demonstrated that *MN* is one-third of *AB.*

Since *AB, BC,* and *BF* are in continued proportion, *AC* and *CF* are also in that same ratio, therefore, as *AB* to *BC,* so *AC* is to *CF,* and as *3AB* is to *3BC, so AC is* to *CF* Whatever ratio *3AB* plus *3BC* has to *3CB, AC* has to some smaller line than CF; let this be *CO.* Then by composition and inversion of ratios, *OA* has to *AC* the same ratio that *3AB* plus *6BC* has to *3AB* plus 35C; further, *AC* has to *SN* the same ratio as *3AB* plus *3BC to AB* plus 25C; by equidistance of ratios, therefore, *OA* has to .MS the same ratio as *3AB* plus 65C *to AB* plus *2BC.* But the ratio of 3,45 plus *6BC to AB* plus *2BC* is *3(AB* plus 25C), therefore *AO* is triple *SN*.

Next, since *OC* is to *CA* as 3C6 is to *3AB* plus 3C5, while as *CA* is to *CF,* so *3AB* is to 35C, then by equidistance of ratios in perturbed proportion, as *OC* is to *CF,* so 3,45 will be to *3AB* plus 35C; and by inversion of ratios, as *OF* is to *FC,* so *3BC* is to *3AB* plus 35C. Also, as *CF* is to *FB, so AC is* to C5, and *3AC* is to 35C; therefore, by equidistance of ratios in perturbed proportion, as *OF* is to *FB,* so *3AC* is to *3(AB* plus 5C). Hence all *OB* will be to *BF* as 6/15 is to *3(AB* plus –BC), and since *FC* has the same ratio to *CA* that CB has to *BA,* then as *FC* is to C4, so *BC* will be to .6,4, and by composition, as *FA* is to *AC,* so is the sum of *BA* plus *AC* to 5,4, as likewise [are] their triples. Therefore, as *FA* is to *AC,* so *3BA* plus 35C is to *3AB;* whence as *FA* is to two-thirds *AC,* so 3,6,4 plus 35C is to two-thirds of *3BA,* which is *2BA.* But as *FA* is to two-thirds *AC,* so F5 is to *MS,* therefore as *FB* is to *MS,* so 35,4 plus *3BC* is to 25,4. But as *OB* is to *FB,* so 6/15 was to *3(AB* plus 5C). Therefore, by equidistance of ratios, *OB* has to *MS* the same ratio as *6AB* to 25/1, whence *MS* is one-third 05. And it was shown that *SN* is one-third *AO,* hence it is clear that *MN* is likewise one-third *AB.* Q. E. D.

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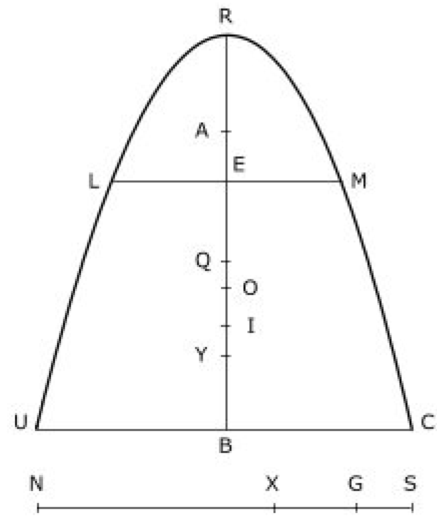
[PROPOSITION 5]

The center of gravity of any frustum cut from a parabolic conoid lies in the straight line that is the axis of this frustum this being divided into three equal parts, the [*said*] *center of gravity lies in the middle* [*part*] *and so divides this* [*part*] *that the portion toward the smaller base has, to the portion toward the larger base, the same ratio as that of the larger base to the smaller*.

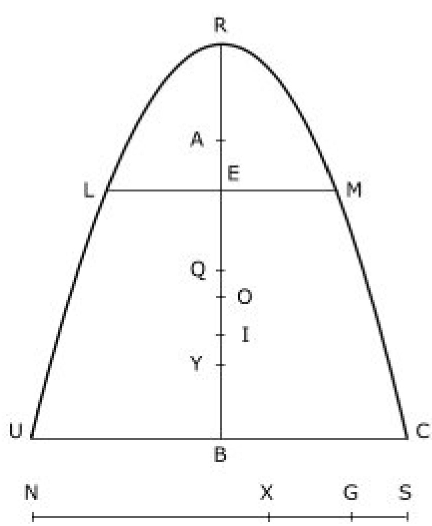
From a conoid whose axis is *RB,* cut a solid with axis *BE,* the cutting plane being parallel to the base. Let it be cut also by another plane, perpendicular to the base, this section giving the parabola *URC,* the sections of the cutting plane and of the base being the straight lines *LM* and *UC.* The diameter of ratios, or parallel diameter, will be *RB*, while *LM* and *UC* will be ordinately applied.[4]

4. Galileo's "diameter of ratios" in the diagram would now be called the axis of ordinates, while his "ordinates" are our abscissae.

Let the line *EB* be divided into three equal parts, of which the middle one is *QY;* this is further divided at *I* so that whatever ratio the base of diameter *UC* has to the base of diameter *LM* (that is, [the ratio] of the square of *UC* to the square of *LM), Ql* has also to *IY* It is to be demonstrated that the center of gravity of the frustum *ULMC* is *I.*



Draw *NS* equal to *BR,* and let *SX* be equal to *ER,* and to *NS* and *SX* take the third proportional *SG,* and as *NG* is to *GS,* let *BQ* be to *IO.* It does not matter whether point *O* falls above or below *LM* And since in section *URC* the lines *LM* and *UC* are ordinately applied, as the square of *UC* is to the square of *LM,* so line *BR* will be to *RE;* and further as the square of *UC* to the square of *LM,* so is *Ql* to *IY,* and as *BR* is to *RE,* so is *NS* to *SX;* therefore *Ql* is to *IY* as *NS* is to *SX* Whence as *QY* is to *YI,* so will *NS* plus *SX* be to *SX,* and as *EB* is to *YI,* so is triple *NS* plus triple *SX* to *SX* Further, as *EB* is to *BY,* so triple the sum of *NS* and *SX* is to the sum of *NS* and *SX;* therefore as *EB* is to *BI,* so is triple *NS* plus triple *SX* to *NS* plus double *SX* Therefore the three lines *NS, SX,* and *GS* are in continued proportion, and whatever the ratio of *SG* to *GN,* the same will be that of some assigned line *OI* to two-thirds of *EB* (that is, of *NX),* and whatever ratio *NS* plus double *SX* has to triple *NS* plus triple *SX,* the same will be that of some assigned line {272} *IB* to *BE* (that is, to *NX).* Therefore, by what was demonstrated above, these [assigned] lines taken together will be one-third of *NS* (that is, of *RB).* Therefore *RB* is triple *BO,* whence *BO* will be the center of gravity of the conoid *URC.*



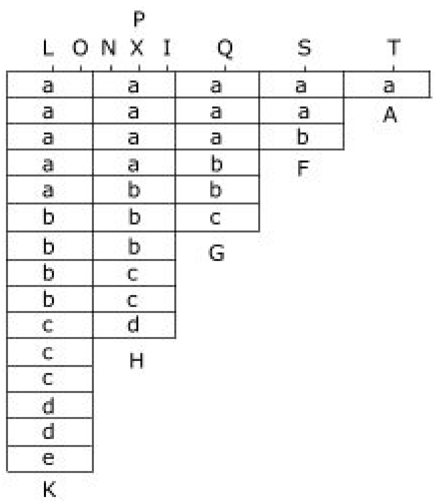
Now let *A* be the center of gravity of the conoid *LRM,* then the center of gravity of the frustum *ULMC* lies in line *OB,* and at the point where this terminates so that whatever ratio the frustum *ULMC* has to the portion *LRM,* the line *AO* has that same ratio to the intercept between *O* and the said point [of termination]. Since *RO* is two-thirds of *RB, RA* is two-thirds of *RE,* and the remainder *AO* will be two-thirds the remainder *EB.* And since as the frustum *ULMC* is to the portion *LRM,* so *NG* is to *GS,* and as *NG* is to *GS,* so is two-thirds *EB* to *OI,* and two-thirds *EB* is equal to line *AO,* then as the frustum *ULMC* is to the portion *LRM, so AO is* to *OI*

4. Galileo's "diameter of ratios" in the diagram would now be called the axis of ordinates, while his "ordinates" are our abscissae.

Therefore it is clear that the center of gravity of the frustum *ULMC* is point /, and the axis is so divided [by it] that the part toward the smaller base is to the part toward the larger base as double the larger base plus the smaller is to double the smaller plus the larger. Which is the proposition, but more elegantly expressed.

[LEMMA]

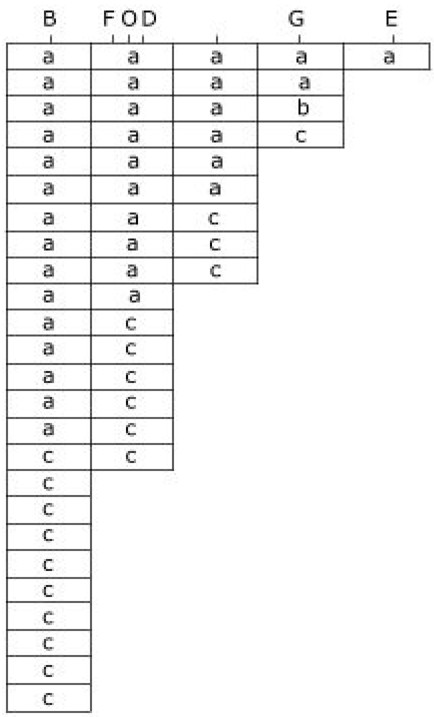
If any number of magnitudes are so arranged that the second adds to the first double the first, and the third adds to the second triple the first, while the fourth adds to the third quadruple the first and so every following magnitude exceeds the preceding one by a multiple of the first magnitude according to its number in order if I say such magnitudes are arranged on a balance and suspended at equal distances, then the center of equilibrium of the whole composite divides that balance so that the part toward the smaller magnitudes is triple the remainder



Let *LT* be the balance, and the magnitudes hanging from it, of the kind described, are *A, F G H, K,* of which *A* is hung first, from *T* I say that the center of equilibrium cuts the balance *TL so* that the part toward Tis triple the remainder Let *TL* be triple *LI,* and *SL* triple *LP,* and *QL* [triple] *LN,* and *LP* [triple] *LO,* then *IP PN NO OL* will be equal. Take at F a magnitude of *2A,* and at *G* another, *3A,* at *H, 4A,* and so on, and let these be the magnitudes [marked] *a* in the diagram. And do the same in magnitudes *F G H K;* {273} indeed, let the magnitude in the remainder of *F,* which is *b,* be equal to *a*, and in *G* take *2b,* in *H, 3b,* etc., and let these be the magnitudes containing 6's. And in the same way take those containing c's, *d's*, and *e.* Then all those in which *a* is marked are equal to [all in] *K;* the composite of all *b's* will equal //, that of the c's, *G,* that composed of all *d's* will be equal to *F,* and *e* [will equal] *A* itself. And since *TI* is double *LI,* / will be the point of equilibrium of magnitudes made up of all the a's, likewise, since *SP* is double *PL, P* will be the point of equilibrium of the composite of all the i's, and for the same cause, *N* will be the point of equilibrium of the composite of all c's, *O* [will be that] of the composite of *d's*, and *L, of e* itself.

There is thus a certain balance *TL* on which at equal distances there hang certain magnitudes *K, H, G F, A,* and further, there is another balance *LI* on which at equal distances hang a like number of magnitudes, equal to and in the same order as those described. Indeed, there is a composite of all a's that hangs from /, equal to *K* hanging from *L,* and a composite of all 6's that hangs from *P,* equal to *H* hanging from *P;* and likewise a composite of c's that hangs from *N*. equal to *G,* and a composite of *el's* that hangs from *O,* equal to *F;* and *e,* hanging from L, is equal to *A.* Whence the balances are divided in the same ratio by the center of [equilibrium of] the composites of magnitudes. But there is [only] one center of the composites of the said magnitudes, and it will be a common point of the line *TL* and the line *LI*. Let this be *X*. And thus as *TX* is to *XL*, so *LX* will be to *XI*, and all *TL* to *LI*. But *TL* is triple *LI*, whence *TX* is triple *XL*.

[LEMMA]



If any number of magnitudes are taken, and the second adds above the first, triple the first, while the third exceeds the second by five times the first, and the fourth exceeds the third by seven times the first, and so on, each addition over the preceding being a multiple of the first according to the successive odd numbers (as the squares of lines that equally exceed one another and of which the excess is equal to the first thereof), and if these be hung at equal distances along a balance, then the center of equilibrium of all combined will divide the balance so that the part toward the lesser magnitudes is more than triple the remainder, but one distance being removed, it will be less than triple.

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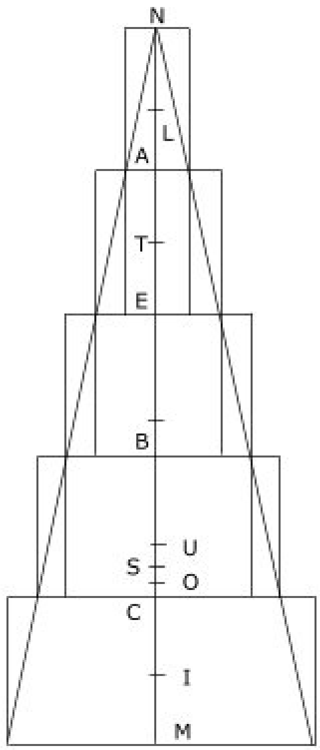
Let there be on the balance *BE* magnitudes such as those described, from which let then be removed some magnitudes arranged among themselves as in the preceding [lemma], let [for example] all the a's [in the present diagram be taken away]. The remainder will be the c's, [still] arranged in the same order [as was the whole], but wanting the greatest [magnitude]. Let *ED* be triple *DB*, and *GF* triple *FB; D* will be the center of equilibrium of everything composed of the a's, while *F* will be that of the c's, hence the center of the compound of [both] a's and c's falls between *D* and *F* And thus it is manifest that *EO* is more than triple *OB*, while *GO* is less than triple *OB*. Which was to be proved.

[PROPOSITION 6]

If any cone or portion of a cone has one figure of cylinders of equal height inscribed to it, and another circumscribed, and if its axis is divided so that the part intercepted between the point of division and the vertex is triple the remainder then the center of gravity of the inscribed figure will be closer to the base of the cone than [*will*] *the point of division, but the center of gravity of the circumscribed* [*figure*] *will be closer to the vertex than* [*will*] *that same point.*



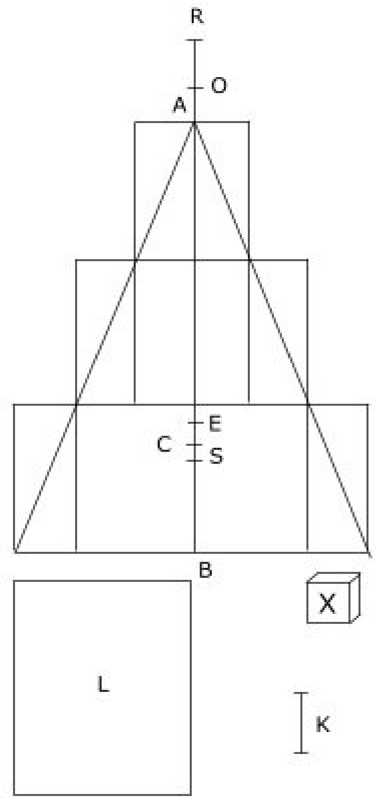
Let there be a cone with axis *NM*, divided at *S* so that *NS* is triple the remainder *SM*, I say that any figure as described that is inscribed in the cone has its center of gravity in the axis *NM*, and that it approaches more nearly the base of the cone than does the point *S*, while the center of gravity of One circumscribed is likewise in the axis *NM*, but closer to the vertex than is S.

Assume an inscribed figure of cylinders whose axes *MC, CB, BE, EA* are equal. Thus this first cylinder, of which the axis is MC, has, to the cylinder with axis *CB*, the same ratio as [that of] its base to the base of the other (since their altitudes are equal), and this ratio is the same as that which the square of *CN* has to the square of *NB*. It is likewise shown that the cylinder with axis *CB* has to the cylinder with axis *BE* the same ratio as that of the square of *BN* to the square of *NE;* while the cylinder around axis *BE* has to the cylinder around axis *EA* the ratio of the square of *EN* to the square of *NA*. Moreover, the lines *NC, NB, EN, NA* equally exceed one another, and their excess is equal to the least, namely *NA*. There are therefore magnitudes (i.e. the inscribed cylinders) {275} which have successively to one another the ratio of squared lines equally exceeding one another, of which the excess is equal to the least. Thus these are arranged on the balance 77, with the single centers of gravity therein, and at equal distances. Hence by those things demonstrated above, it is evident that the center of gravity of all these compounded in the balance *TI* so divides it that the part toward *T* is more than triple the remainder.[5] Let this center be *O*, then *TO* is more than triple *OI* But *TN* is triple *IM*, therefore all *MO* will be less than one-quarter of all *MN*, of which *MS* was assumed to be one-quarter It is therefore evident that point *O* comes nearer the base of the cone than does *S*.

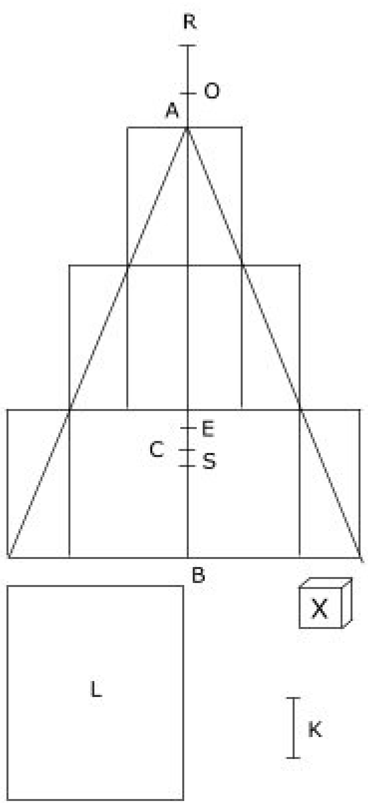
Now let the circumscribed figure consist of cylinders whose axes *MC CB, BE, EA, AN* are equal to one another. As with the inscribed [figure], these are shown to be to one another as the squares of lines *NM NC BN NE A N*, which equally exceed one another and whose excesses equal the least, *AN*. Whence, from what went before, the center of gravity of all the cylinders thus arranged (and let this be *U*) so divides the balance *RI* that the part toward *R* (that is, *RU)* is more than triple the remainder *UI*, while *TU* will be less than triple the same. But *NT* is triple *IM*, therefore all *UM* is greater than one-quarter of all *MN*, of which *MS* was assumed to be one-quarter. And thus point *U* is closer to the vertex than is point *S*. Q.E.D.

[PROPOSITION 7]

Given a cone, a figure can be inscribed and another circumscribed to it, made up of cylinders having equal heights, so that the line intercepted between the center of gravity of the circumscribed [*figure*] *and that of the inscribed* [*figure*] *is less than any assigned line.*



Given a cone with axis *AB,* and given further a straight line *K,* I say, let the cylinder *L* be drawn equal to that [which may be] inscribed in the cone, having an altitude of one-half the axis *AB.* Divide *AB* at *C* so that *AC* is triple *CB;* and whatever ratio *AC* has to *K*, let this cylinder *L* have to some solid, *X* Circumscribe about the cone a figure of cylinders having equal altitudes, and inscribe another one, so that the circumscribed exceeds the inscribed by a quantity less than the solid *X* Let the center of gravity of tthe circumscribed {276} the solid *X* Let the center of gravity of the circumscribed [figure] be *E*, which falls above *C*, while the center of the inscribed one is *S,* falling below *C.* I now say that line *ES* is less than *K.*



For if it is not, put *CA* equal to *EO*, then since *OE* has to *K* the same ratio as that of *L* to *X*, the inscribed figure is not less than cylinder *L,* and the excess by which the circumscribed figure surpasses it is less than solid *X;* therefore the inscribed figure has to the said excess a greater ratio than *OE* will have to *K.* But the ratio of *OE* to *K* is not less than that of *OE* to *ES,* since *ES* cannot be assumed less than *K',* therefore the inscribed figure has a greater ratio to the excess by which the circumscribed [figure] surpasses it than *OE* has to *ES.* Hence whatever ratio the inscribed [figure] has to the said excess, some line greater than *EO* will have this to the line *ES.* Let this [line] be *ER.* Now, the center of gravity of the inscribed figure is *S*, while that of the circumscribed is *E;* hence it is evident that the remaining portions by which the circumscribed exceeds the inscribed [figure] have their center of gravity in line *RE*, and at that point where it is terminated so

5. Because *TI* omits one distance, *NA.*

that whatever ratio the inscribed [figure] has to those portions, the line intercepted between *E* and that point has to line *ES*. But *RE* has this ratio to *ES;* hence the center of gravity of the remaining portions by which the circumscribed figure exceeds the inscribed will be *R,* which is impossible, since indeed the plane through *R* [drawn] parallel to the base of the cone does not cut these portions. Therefore it is false that line *ES* is not less than *K,* and hence it will be less.

Moreover, in a way not dissimilar, this may be demonstrated to hold for pyramids. From this it is manifest that:

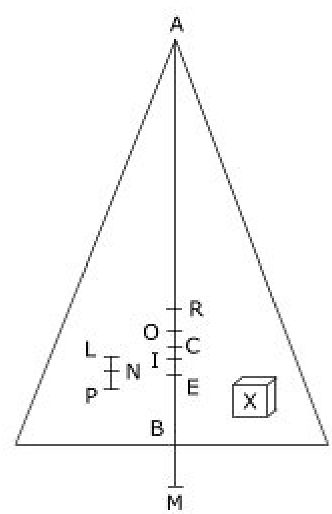
[COROLLARY]

About a given cone, a figure can be circumscribed, and [*within it*] *another inscribed, of cylinders having equal altitudes, such that the lines between their centers of gravity and the point which divides the axis of the cone so that the part toward the vertex is triple the remainder are less than any given line.*

For indeed, as was demonstrated, the said point dividing the axis in the said way is always found between the centers of gravity of the circumscribed and inscribed [figures], and it is possible for the line between those same centers to be {277} less than any assigned line, so that which is intercepted between either of the two centers and the point that thus divides the axis must be much less than this assigned line.

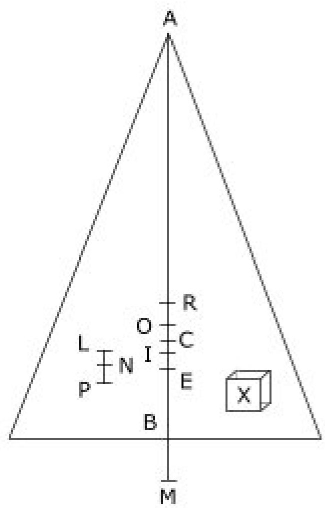
[PROPOSITION 8]

The center of gravity of any cone or pyramid so divides the axis that the part toward the vertex is triple the remainder toward the base.



Given the cone with axis *AB,* divided so that *AC* is triple the remainder *CB,* it is to be shown that C is the center of gravity of the cone. For if it is not, the center of the cone will be either above or below point *C*. First let it be below, at *E,* and draw line *LP* equal to *CE*, and divide this anywhere at *N*, and whatever ratio *BE* plus *PN* shall have to *PN,* let this cone have to some solid, *X* Inscribe in the cone a solid figure made up of cylinders of equal height, the center of gravity of this shall be less distant from point *C* than [the length of] line LTV, and the excess by which the cone exceeds [this figure] will be less than solid *X* It is clear from what has been demonstrated that these things can be done. Let this solid figure which we assume have its center of gravity at / Then line *IE* will be greater than *NP*, since *LP* is equal to *CE;* and *1C* [is] less than *LN*, and since *BE* plus *NP* is to *NP* as the cone is to *X,* and moreover the excess by which the cone exceeds the inscribed figure is less than solid *X,* the cone will have a greater ratio to the said excess than that of *BE* plus *NP* to *NP,* and by division, the inscribed figure has a greater ratio to the excess by which the cone exceeds it than *BE* has to *NP* Moreover, *BE* has to *El* a still smaller ratio than it has to *NP,* since *IE* is greater than *NP,* whence the inscribed figure has a much greater ratio to the excess by which the cone surpasses it than *BE* has to *El.*

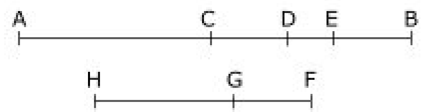
Therefore whatever ratio the inscribed [figure] has to the said excess, some greater line *BE* has to line *El.* Let this be *ME;* since *ME* is to *El* as the inscribed figure is to the excess by which the cone surpasses it, and [if] *E* is the center of gravity of the cone, while / is the center of gravity of the [figure] inscribed, then *M* will be the center of gravity of the remaining portions by which the cone exceeds the inscribed figure in it, which is impossible. Therefore the center of gravity of the cone is not below point *C.* {278}



But neither is it above. For, if possible, let it be *R,* again take the line *LP,* cut anywhere at *N* Whatever ratio *BC* plus *NP* has to *NL,* let the cone have to *X,* and likewise circumscribe about the cone a figure that exceeds it by a lesser quantity than the solid *X;* the line intercepted between its center of gravity and *C* shall be less than *NP* Now let there be circumscribed [a figure] having center of gravity *O,* the remainder *OR* will be greater than *NL.* And since as *BC* plus *PN* is to *NL,* so the cone is to *X,* but the excess by which the circumscribed [figure] surpasses the cone is less than *X,* and *BO* is less than *BC* plus *PN,* while *OR* is greater than *LN,* the cone will have a greater ratio to the remaining portions by which it is exceeded by the circumscribed figure than *BO* has to *OR.* Let *MO* have that ratio to *OR,* then *MO* will be greater than *BC,* and M will be the center of gravity of the portions by which the cone is exceeded by the circumscribed figure, which is contradictory Therefore the center of gravity of this cone is not above the point *C,* but neither is it below, as was shown, therefore it is *C* itself. And the same may be demonstrated in the above way for any pyramid.

[LEMMA][6]

If there are four lines in [*continued*] *proportion, and whatever ratio the least of these has to the excess by which the greatest exceeds the least, that same* [*ratio*] *is had by some* [*assumed*] *line to 3/4 of the excess by which the greatest exceeds the second* [*line*] *and whatever ratio a line equal to the greatest plus double the second plus triple the third has to a line equal to four times the sum of the greatest, the second, and the third together that same ratio is had by* [*another*] *assumed line to the excess by which the greatest exceeds the second and these two* [*assumed*] *lines taken together will be one-quarter of the greatest of the original lines.*



Let there be four lines in continued proportion, *AB*, *BC BD BE'*, and whatever ratio *BE* has to *EA*, let *FG* have t three-quarters of *AC'*, and further, whatever ratio a line equal to *AB* plus *2BC* plus *3BD* has to a line equal to four times the sum of *AB*, *BC*, and *BD*, let *HG* have to *AC*. It is to be shown that *HF* is one-quarter of *AB*.

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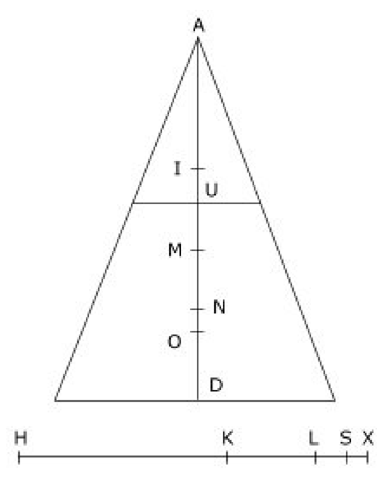
Since *AB*. *BC BD*, and *BE* are proportional, then *AC*, *CD*, and *DE* will be in that same ratio, and as four times the sum of *AB*, *BC*, and *BD* is to *AB* plus *2BC* plus *3BD*, so the quadruple of *AC* plus *CD* plus *DE* (that is, *4AE)* is to *AC* plus 2CD plus 3*DE*; and thus is *AC* to *HG*. Therefore as *3AE* is to *AC* plus 2*CD* plus *3DE*, so is three-quarters of *AC* to *HG*. Moreover, as *3AE* is to 3*EB*, so is three-quarters of *AC* to *GF*. Hence, by the converse of [Euclid] V, 24, as *3AE* is to *AC* plus *2CD* plus 3/3.6, so is three-quarters of *AC* to *HF*; and as *4AE* is to *AC* plus 2*CD* plus 3*DB* (that is, to *AB* plus *CB* plus *BD*), so *AC* is to *HF*. And permuting, as 4*AE* is to *AC,* so *AB* plus *CB* plus *BD* is to *HF*. Further, as *AC* is to *AE*, so *AB* is to *AB* plus *CB* plus *BD*. Hence, by equidistance of ratios in perturbed proportion, as *4AE* is to *AE*, so *AB* is to *HF*. Whence it is clear that *HF* is one-quarter of *AB*.

[PROPOSITION 9]

Any frustum of a pyramid or cone cut by a plane parallel to its base has its center of gravity in the axis, and this so divides it that the part toward the smaller base is to the remainder as three times the greater base plus double the mean proportional between the greater and smaller bases plus the smaller base is to triple the smaller base plus the said double of the mean proportional distance plus the greater base.

From a cone or pyramid with axis *AD*, cut a frustum by a plane parallel to the base having axis *UD*, and whatever ratio triple the larger base, plus double the mean proportional [of both bases] plus the smaller [base], has to triple the smaller, plus double the [above] mean proportional plus the greatest, let *UO* have to *OD*. It is to be shown that *O* is the center of gravity of the frustum.



Let *UM* be one-quarter of *UD*. Draw line *HK* equal to *AD*, and let *KX* equal *AU*, let *XL* be the third proportional to *HX* and *KX*, while *XS* is the fourth proportional. Whatever ratio *HS* has to *SX*, let *MD* have to a line from *O* in the direction of *A*, and let this be *ON* Now since the

6. A manuscript copy submitted in 1587 (note 1, above) exhibits some variants from the printed text, but none of a substantial character.

larger base is to the mean proportional between the larger and the smaller as *DA* is to *AU* (that is, as *HX* is to *XK),* and the said mean proportional is to the smaller as *KX* is to *XL,* then the larger, the mean proportional, and the smaller base will be in the ratio of lines *HX XK,* and *XL.*

Thus as triple the larger base plus double the mean {280} proportional plus the smaller is to triple the smaller plus double the mean proportional plus the larger (that is, as *UO* is to *OD),* so is triple *HX* plus double *XK plus XL* to triple *XL* plus double *XK* plus *XH* And, by composition and inverting, *OD* will be to DC/as *HX* plus double *XK plus* triple *XL* is to four times the sum of *HX XK,* and *XL.* Therefore there are four lines in continued proportion, *HX XK, XL,* and *XS',* and whatever ratio *XS* has to *SH,* some assumed line *NO* has to three-quarters of *DU* (that is, to three-quarters of *HK).* Further, whatever ratio *HX* plus double *XK* plus triple *XL* has to four times the sum of *HX XK,* and *XL,* some assumed line *OD* has to *DU (that* is, to *HK).* Hence, by what was demonstrated, *DN* will be one-quarter of *HX* (that is, of *AD),* whence point *N* will be the center of gravity of the cone or pyramid having axis *AD.*

Let *I* be the center of gravity of the pyramid or cone having axis *AU.* It is then clear that the center of gravity of the frustum lies in line *IN* extended beyond *N,* and at that point of it which, with point N, intercepts a line to which *IN* has the ratio that the frustum cut off has to the pyramid or cone having axis *A U* . Thus it remains to be shown that *IN* has to *NO* the same ratio that the frustum has to the cone whose axis is *A U* But as the cone with axis *DA* is to the cone with axis *A U,* so is the cube of *DA* to the cube of *A U,* that is, as the cube of *HX* to the cube of *XK ;* and this is the ratio of *HX* to *XS.* Whence, dividing, as *HS* is to *SX,* so the frustum having axis *DU* will be to the cone or pyramid having axis *UA.* And as *HS* is to *SX,* so also *MD* is to *ON,* whence the frustum is to the pyramid having axis *A U* as *MD* is to *NO.* And since *AN* is three-quarters of *AD,* and *AI* is three-quarters of *A U,* the remainder *IN* will be three-quarters of the remainder *UD,* wherefore *IN* will be equal to *MD.* It was demonstrated that *MD* is to *NO* as the frustum is to the cone *A U;* therefore it is clear that *IN* has also this same ratio to *NO.* Whence the proposition is clear.

Finis[7]

7. The end of the original printed edition.

Galileo Galilei, *Discourses and Mathematical Demonstrations Concerning Two New Sciences Pertaining to Mechanics and Local Motions.* Appendix translated by Stillman Drake, University of Wisconsin Press, Madison, 1974: 261–280. Additional selection from this source:

THE ADDED (OR "FIFTH" DAY) BY STILLMAN DRAKE (1974)