## INTIIMATIONS <br> OF <br> COMMONALITTY $\mathbb{I N}$ <br> PLANETARY <br> SYSTEMS



2023

## 1611'-"A New Year's Gift of Hexagonal Snow" - $2023^{2}$



1 "Strena Seu de Nive Sexangula." (Johannes Kepler's 1611 treatise The Six-Cornered Snowflake)
${ }^{2}$ Spirasolaris.ca/2023-2024

# INTIMATIONS OF COMMONALITY IN PLANETARY SYSTEMS 

## PART I.

The inclusion of mean synodic periods between adjacent planets completes a stalled 19th Century model of the Solar System developed by American Benjamin Pierce (1809-1880). The model suggests that the Solar System may have been subject to disruptions in the past which include the Mars-Jupiter gap, and also that at present Earth occupies an intermediate, synodic location between Venus and Mars, albeit not precisely. (Figure 1). The resulting framework also suggests that complex elements of the Fibonacci series, Lucas series and Phi-series underlie the dynamic structure of the Solar System. As a result, with twice the number of planetary periods now available, the incorporation of these three series and synodic cycles permits development of a working planetary framework for general testing. A search for further enlightenment leads to Babylonian astronomy and a wider ranging series of of inquiries with positive and negative results sufficient enough to merit the inclusion an excursus at the conclusion of Part V .

## PART II.

The revised framework was applied to a variety of external planetary systems with similarities encountered (e.g., in the structure of HR 8799) explained in part by synodic relations common to the Solar System inherent in the Phi-series. Other similarities between HR 8799 and the Solar System include the possible demise of the fifth planet and possible outward shifts by both HR 8799d and HR 8799e to intermediate, synodic locations. A theoretical inward extension for this system results in a period of 0.2405942 years (HR 8799_9) versus 0.24084445 years for the Solar System's Mercury. (Figs. 1 \& 2).

## PART III.

Further tests lead to real-time planetary motion in the Solar System with results which confirm pheidian similarities already encountered with the mean periods in Part I including historical aspects explored at length in the optional excursus. Further concerns regarding possible 795 -year cycles for the four major superior planets gives rise to the suspicion that such matters may have significant implications while phyllotaxis in such contexts is also considered. (Figs.1-6e).

## PART IV.

Again searching for enlightenment Part IV deals with the two triangles in Plato's Timaeus, the isosceles and the equilateral triangles. It is shown the former pertains to the Fibonacci series, the latter to the Lucas series and both with respect to the "Rotation of the Elements."
The results are again pheidian and again apply to planetary frameworks. (Tables $1 \& 2$ ).

## PARTV.

The spiral form in time and place remains pheidian in both form and interest while initially concentrating on ammonites, where David Raup's neglected researches are re-examined and re-applied to ammonites after developing a series of double-precision pheidian test spirals. Later the study widens to include radiolarians and diodoms, but ends with concerns which are not so much about the Solar System and exoplanetary systems per se - but ourselves, our past, our present and our increasingly uncertain future.

With this in mind, a gentle, non-destructive option is suggested for those who might wish to embrace it.

## PART ONE



THE PIERCE PLANETARY FRAMEWORK (1850) REVISITED

## INTRODUCTION

In the 1850s American scientist Benjamin Peirce (1809-1880) produced a robust heliocentric planetary framework by applying Fibonacci-based reduction ratios to the mean periods of revolution of the eight Solar System planets. ${ }^{1}$ Partially incomplete in dynamic terms and subjected to alternative viewpoints, this promising approach was oddly dismissed despite the attendant ramifications and total absence of any comparable planetary theory. Fortunately, however, a condensed version was at least preserved by Louis Agassiz in the latter's Essay on Classification (1859). ${ }^{2}$

As described in the latter work, Peirce began by assigning the outermost planet Neptune a convenient (albeit high) mean period of revolution of 62,000 days. Next, moving inwards, planetary periods rounded to the nearest day were derived from planet-to-planet reduction factors formed from Fibonacci numbers (1, 1, 2, 3, 5, 8, 13, 21, 34, $55,89,144,233$, etc.), specifically, successive alternate Fibonacci ratios of $1 / 2,1 / 3,2 / 5,3 / 8,5 / 13,8 / 21,13 / 34$ and 21/55. Thus the 62,000-day period of Neptune was reduced by one-half to obtain a mean period of revolution for Uranus of 31,000 days followed by a one-third reduction of the latter to produce a 10,333-day period for Saturn, 4,133 days for Jupiter (2/5), and so on down to an 87-day period for the innermost planet Mercury from a final reduction ratio of $13 / 34$. However, despite this encouraging end-to-end correspondence a reduction factor for Earth was entirely absent from the alternate Fibonacci sequences. In fact, the inclusion of the latter required two additional reduction ratios of $8 / 13$ and $13 / 21$. The last ratio in Pierce's original list ( $21 / 55$ ) remained unused but was most likely included for continuity and support for the latter's contention that "There can be no planet planet exterior to Neptune, but there may be one interior to Mercury." ${ }^{3}$

The Fibonacci-based reduction ratios, resulting periods and comparison with $19^{\text {th }}$ Century Solar System periods were published in the Essay on Classification in two sparse, unlabelled tables ${ }^{4}$ based on subdivisions of the 62,000day period for Neptune. The initial results are shown in Table 1a with title and column assignments added:
$\left.\begin{array}{lcrr}\hline \hline \begin{array}{l}\text { PLANETS } \\ \text { (ca. 1850) }\end{array} & \begin{array}{c}\text { PERIODS } \\ \text { actual/days }\end{array} & \begin{array}{r}\text { PERIODS } \\ \text { (reductions) }\end{array} & \begin{array}{r}\text { RATIOS } \\ \text { (Pierce) }\end{array} \\ \hline \text { Neptune, } & 60,129 & 62,000 & \\ \text { Uranus, } & 30,687 & 31,000 & 1 / 2 \\ \text { Saturn, } & 10,759 & 10,333 & 1 / 3 \\ \text { Jupiter, } & 4,333 & 4,133 & 2 / 5 \\ \text { Asteriods, } & 1,200 \text { to } 2,000 & 1,550 & 3 / 8 \\ \text { Mars, } & 687 & 596 & 8 / 13 \\ \text { Earth, } & 365 & 366 & 8 / 13 \\ \text { Venus, } & 225 & 227 & 13 / 21\end{array}\right\} 8 / 21$

Table1a. The initial planetary structure, Peirce (1852:129)
Next, the planetary framework was extended to include twinned ratios provided by adjacent Fibonacci numbers. This produced the same periods of revolution for the planets plus intermediate periods on either side with Earth in an intermediate location between Mars and Venus. Pierce included the intermediate positions for comparable $19^{\text {th }}$ Century data in the fourth column, but apart from 365 days for Earth no other intermediate periods were given. The final ratios and reductions are shown in Table 1b, again with the title and column assignments added:

| $\xrightarrow{\text { PLANETS }}$ | RATIOS (Pierce) | PERIODS I <br> (reductions) | PERIODS II <br> (actual/days) |
| :---: | :---: | :---: | :---: |
| Neptune | 1/1 | 62,000 | 60,129 |
|  | 1/1 | 62,000 |  |
| Uranus | 1/2 | 31,000 | 30,687 |
|  | 1/2 | 15,500 |  |
| Saturn | 2/3 | 10,333 | 10,759 |
|  | 2/3 | 6,889 |  |
| Jupiter | 3/5 | 4,133 | 4,333 |
|  | 3/5 | 2,480 |  |
| Asteriods, | 5/8 | 1,550 | 1,200 |
| Mars | $5 / 8$ $8 / 13$ | 968 596 | 687 |
| Earth | 8/13 | 366 | 365 |
| Venus | 13/21 | 227 | 225 |
| Mercury | $13 / 21$ $21 / 34$ | 140 87 | 88 |

Table1b. The Final planetary structure, Peirce (1852:129)

The final framework languished in this unfinished form despite correlations which included the Mars-Jupiter gap plus the possibility that planet Earth may, perhaps, be occupying an intermediate location. This troubling indicator should surely have been investigated, beginning, one might suggest, with mean synodic motion in general and mean synodic lap-cycles in particular.

## Mean synodic motion and the intermediate periods

In fact, all of the intermediate intervals introduced by Pierce are the mean synodic periods between adjacent planets. In other words, lap-cycle times faster-moving inner planets require to complete $360^{\circ}$ of direct orbital motion with respect to that of slower-moving outer planets. Adjacent or otherwise, mean synodic periods $(S)$ between planets with mean periods of revolution $T_{1}$ and $T_{3}$ are derived from the lesser used general synodic formula:

$$
\begin{equation*}
\text { Synodic period } S_{2}=\frac{T_{1} \cdot T_{3}}{T_{1}-T_{3}}\left(T_{1}>S_{2}>T_{3}\right) \tag{1}
\end{equation*}
$$

although in modern practice relation (1) is rarely applied in this form. Synodic periods in planetary tables normally pertain to either the lap-cycles of Earth with respect to the slower outer (superior) planets or the lap-cycles of the faster inner (inferior) planets with respect to Earth itself. In both cases, with the reference period of Earth exactly one year, redundant multiplications by unity are unstated, resulting in the standard synodic formulas:

$$
\begin{equation*}
\text { Superior planets } S_{\mathrm{s}}=\frac{T_{\mathrm{s}}}{T_{\mathrm{s}}-1} \quad \text { (1s) } \quad \text { Inferior planets } S_{\mathrm{i}}=\frac{T_{\mathrm{i}}}{1-T_{\mathrm{i}}} \tag{1i}
\end{equation*}
$$

Nevertheless, relation (1) is more useful in the present context, as is relation (2), where, with both the outer period $T_{1}$ and intermediate period $S\left(=T_{2}\right)$ known, the innermost period $T_{3}$ can be obtained from:

$$
\begin{equation*}
\text { Inner period } T_{3}=\frac{T_{1} \cdot T_{2}}{T_{1}+T_{2}}\left(T_{1}>T_{2}>T_{3}\right) \tag{2}
\end{equation*}
$$

Relation (1) permits the restoration of the missing intermediate periods in Table 1b, and allied with relation (2) plus period formulas employing geometric means - relations (4) and (4E) introduced later - all have roles to play in tests on external planetary systems that follow. More immediately, with missing synodic periods supplied and dynamic component incorporated, a standard planetary framework predicated on Peirce's Fibonacci-based approach can now be assembled as follows.

## Units of time and measure

Standard years with respect to unity and also the Julian year of 365.25 days are applied in the present study, the first for comparison with modern periods in Julian years, ${ }^{5}$ and the second for real-time calculations of planetary motion in Part Three utilising the methodology developed by Bretagnon and Simon (1986). ${ }^{6}$

## Standard order, positions and titles

Following the order adopted by Pierce, the mean periods of revolution and the mean synodic intervals have been assigned standard position numbers and uniform titles commencing with the first and outermost planet. Thus for the eight-planet Solar System the relative synodic period (or lap-cycle) of Planet \#2 (Uranus) with respect to that of outermost Planet \#1 (Neptune) is Synodic 2-1 followed by Synodic 3-2 between Planet \#3 (Saturn) and Planet \#2 (Uranus), and so on, down to Synodic 8-7 between innermost planet Mercury (\#8) and Planet \#7 Venus. Planetary positions interior to Mercury (Intra-Mercurial-Objects, or IMOs) commence at IMO 1 followed by IMO 2, etc., with the intermediate synodic periods, Pierce reduction ratios and later divisors continuing inwards in due order. In this theoretical framework, Earth (with reservations) occupies the Synodic 7-6 location between \#6 Mars and \#7 Venus.

## Divisors for the sequential periods of revolution and intermediate synodic intervals

Next, the awkward multiplications by successive reduction factors used by Pierce are replaced by a standard set of divisors applied to the base period alone, a practice already in use for exoplanets. Thus for the eight-planet Solar System the standard integer divisors for the periods of revolution of the planets beginning with the outermost (\#1) are: 1, 2, 6, 15, 40, 104, 273, 714. Divisors for the intermediate mean synodic periods (lap-cycles) are in turn: $1,4,9,25,64,169,441$, thus the synodic divisors are all sequential squares of the Fibonacci Series.

The complete set of divisors with intermediate synodic divisors shown in brackets is therefore: $\mathbf{1 , ( 1 )} \mathbf{2},(4) \mathbf{6},(9)$ 15, (25) 40, (64) 104, (169) 273, (441) 714, plus (1156) 1870, (3025) and 4895 for ten-planet systems, etc.

## Base periods B1, B2, B3, B4 and B5 for the divisors

Although the period of revolution of the outermost planet (base period B1) is of fundamental importance in Pierce's planetary model, the calculated value for Synodic 2-1 is in fact 62,620 days (hereafter base period B2) which exceeds the latter's initial base period of 62,000 days (hereafter, one-off base period P2). Nevertheless, when used as the base period for the divisors, Synodic 2-1 yields marginally superior results compared to those obtained with P2. Therefore, where Synodic 2-1 differs from B1 a second base period (B2) can be added for further testing. Other bases (B3s) can be approximated by applying the planetary divisors in reverse, i.e., as multipliers of known periods with known locations in otherwise incomplete systems. Where advantageous, the mean value (B4) of multiple B3 products and/or a substitute B5 (yielding least errors) may also be applied at the expense of further complexity.

## Resonant triples between planets [RZT]

Resonant triples between planets are included for completeness in Solar System Table 2 and elsewhere. Related to both the twinned Pierce ratios and added divisors, resonant triples are obtained from the bracketing periods of revolution of adjacent planets and the synodic periods in between. Thus, for Neptune and Uranus [1(1)2], Uranus and Saturn [1(2)3], Saturn and Jupiter [2(3)5], etc. Their immediate relevance lies in the fact that the associated divisors are sequential Fibonacci multiples with the central value of each triple providing the multiplication factor. - $1 x$ for the first set: [1(1)2], $2 x$ for the second, thus [2(4)6], $3 x$ for the third [6(9)15], $5 x$ for the fourth [15(25)40], etc.

## Fibonacci Periods in days below Mercury

The resulting Pierce P2 planetary framework for a thirteen-planet extension of the Solar System is shown in Table 2a with intermediate positions for the synodic periods and division of modern periods (Base B2/Divisors) included for comparison. The paired resonances from Unity to the Major Sixes (the reverse of the Pierce reduction ratios) aid the analyses of exoplanetary systems in Part Two while also bringing to mind ancient methodology, e.g., "Music of the Spheres," which, though not music per se, nevertheless appears to have a role in this complex matter. As does the presence of the Fibonacci series below Mercury expressed in days generated by the P2 and the B2 divisors also included in the Table.

| PLANETS N Synodic \# | RATIOS <br> (Pierce) | DIVISORS (added) | RESONANCES (to Major 6's) ${ }^{\text {a }}$ | RES.TRIPLES (to IMO 5) | PERIODS1 P2/Divisors | PERIODS1 T $(J Y R=365.25)$ | DISTANCES1 R (Ref. unity/a.u) | PERIODS2 <br> B2/Divisors |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Neptune 1 | 1/1 | 1 | 1:1 | $1(1) 2$ | 62,000 | 169.74675 | 30.657329 | 62,620 |
| Synodic 2-1 | 1/1 | 1 | $1: 1$ |  | 62,000 | 169.74675 | 30.657329 | 62,620 |
| Uranus 2 | 1/2 | 2 | Octave \#1, $2: 1$ | 1(2)3 | 31,000 | 84.873374 | 19.312907 | 31,310 |
| Synodic 3-2 | 1/2 | 4 | Octave \#2, 4:2 | (2) | 15,500 | 42.436687 | 12.166369 | 15,655 |
| Saturn 3 | 2/3 | 6 | Fifth \#1, 6:4 |  | 10,333 | 28.291125 | 9.2846772 | 10,437 |
| Synodic 4-3 | 2/3 | 9 | Fifth \#2, 9:6 |  | 6,889 | 18.860750 | 7.0855348 | 6,958 |
| Jupiter 4 | 3/5 | 15 | Major 6\#1, 15:9 | 3(5)8 | 4,133 | 11.316450 | 5.0404993 | 4,175 |
| Synodic 5-4 | 3/5 | 25 | Major 6 \#2, 25 : 15 |  | 2,480 | 6.7898700 | 3.5857029 | 2,505 |
| M-J Gap 5 | 5/8 | 40 |  | 5(8)13 | 1,550 | 4.2436687 | 2.6211647 | 1,566 |
| Synodic 6-5 | 5/8 | 64 |  | $5(8) 13$ | 986 | 2.6522930 | 1.9160830 Fi | nacci ${ }^{\text {b }} 978$ |
| Mars 6 | 8/13 | 104 |  | 8(13)21 | 596 | 1.6480267 | 1.3952204 | 610602 |
| Earth/Syn 7-6 | 8/13 | 169 |  |  | 366 | 1.0044186 | 1.0029436 | 377371 |
| Venus 7 | 13/21 | 273 |  | 13(21)34 | 227 | 0.6217830 | 0.7284938 | 233229 |
| Synodic 8-7 | 13/21 | 441 |  |  | 140 | 0.3849133 | 0.5291457 | 144142 |
| Mercury 8 | 21/34 | 714 |  | 21(34)55 | 87 | 0.2377405 | 0.3837681 | 8988 |
| Synodic 9-8 | (21/34) | 1156 |  |  | 54 | 0.1468397 | 0.2783315 | $55 \quad 54$ |
| IMO 19 | (34/55) | 1870 |  | 34(55)89 | 33 | 0.0907737 | 0.2019792 | $34 \quad 33$ |
| Synodic 10-9 | (34/55) | 3025 |  | 34(5)89 | 20 | 0.0561146 | 0.1465719 | 2121 |
| IMO 210 | (55/89) | 4895 |  | 55(89)144 | 13 | 0.0346776 | 0.1063406 | 1313 |
| Synodic 11-10 | (55/89) | 7921 |  | $55(89) 144$ | 8 | 0.0214300 | 0.0771521 | 88 |
| IMO 311 | (89/144) | 12816 |  | 89(144)233 | 5 | 0.0132449 | 0.0559800 | 55 |
| Synodic 12-11 | (89/144) | 20736 |  | 89(144)233 | 3 | 0.0081861 | 0.0406179 | $3 \quad 3$ |
| IMO 412 | (144/233) | 33552 |  |  | 2 | 0.0050592 | 0.0294706 | 22 |
| Synodic 13-12 | (144/233) | 54289 |  | 144(233)377 | 7 | 0.0031267 | 0.0213826 | 11 |
| IMO 513 | (233/377) | 87841 |  |  | 1 | 0.0019324 | 0.0155144 | 11 |

Table2a. The enhanced planetary structure: ratios, divisors, triples, periods in days \& years; P2 distances (a.u.).
${ }^{\text {a }}$ Octave $2: 1$, Fifth $3: 2$, Major Six $5: 3$. ${ }^{\text {b }}$ The extension to Planet 13 concludes at Fibonacci number 1 .

## The Solar System revisited

Table 2 b shows the uniform assignments, the twinned Pierce ratios, added divisors, resonant triples and the results generated by Pierce base P2, followed by modern base periods B2 and B1 with the latter in both Julian years and days. Also included with the two sets of data are the calculated synodic periods, Mars-Jupiter geometric mean between the periods of the latter pair, associated synodic positions on either side, and Earth located in the synodic position between Venus and Mars. The 365.25 day period for Earth is substituted in the second set of modern data although the actual synodic period (Synodic 7-6) is 335 days, thus less than one year and 366 -day period obtained from the P2 ratio and 62,000-day base period. Atypical Venus-Earth and Earth-Mars synodic periods and the Mars-Jupiter synodic cycle are omitted for clarity. The periods in days are rounded; red periods equal exact Fibonacci numbers.

| PLANETS N Synodics \# | RATIOS <br> (Pierce) | DIVISOR <br> (added) | $\begin{aligned} & \text { RES.TRIPLE P } \\ & {[(\text { RZT })]} \end{aligned}$ | 2/Divisors | PERIODS2 Actual/days | MODERN1 B2/Divisors | MODERN2 B1Julian yrs | MODERN (Days) | MODERN2 (Days) |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Neptune 1 | 1/1 | 1 |  | 62,000 | 60,129 | 171.44429 | 163.72320 | 62,620 | 59,800 |
| Synodic 2-1 | 1/1 | 1 | 1(1)2 | 62,000 | 62,672 | 171.44429 | 171.44429 | 62,620 | 62,620 |
| Uranus 2 | 1/2 | 2 |  | 31,000 | 30,687 | 85.722145 | 83.747407 | 31,310 | 30,589 |
| Synodic 3-2 | 1/2 | 4 | 1(2)3 | 15,500 | 16,658 | 42.861072 | 45.360219 | 15,655 | 16,568 |
| Saturn 3 | 2/3 | 6 |  | 10,333 | 10,759 | 28.574048 | 29.423519 | 10,437 | 10,747 |
| Synodic 4-3 | 2/3 | 9 | 2(3)5 | 6,889 | 7,255 | 19.049366 | 19.858872 | 6,958 | 7,253 |
| Jupiter 4 | 3/5 | 15 |  | 4,133 | 4,333 | 11.429619 | 11.856525 | 4,175 | 4,331 |
| Synodic 5-4 | 3/5 | 25 | 3(5)8 | 2,480 | 2,867 | 6.8577717 | 7.8476788 | 2,505 | 2,866 |
| M-J Gap 5 | 5/8 | 40 |  | 1,550 | 1,725 | 4.2861072 | 4.7221497 | 1,556 | 1,725 |
| Synodic 6-5 | 5/8 | 64 | 5(8)13 | 986 | 1,142 | 2.6788170 | 3.1255291 | 978 | 1,142 |
| Mars 6 | 8/13 | 104 |  | 596 | 687 | 1.6485028 | 1.8807111 | 602 | 687 |
| Earth/Syn 7-6 | 8/13 | 169 | 8(13)21 | 366 | 335 | 1.0144633 | 1.0000000 | 371 | 365 |
| Venus 7 | 13/21 | 273 |  | 227 | 225 | 0.6280011 | 0.6151826 | 229 | 225 |
| Synodic 8-7 | 13/21 | 441 | 13(21)34 | 140 | 145 | 0.3887626 | 0.3958008 | 142 | 145 |
| Mercury 8 | 21/34 | 714 |  | 87 | 88 | 0.2401186 | 0.2408445 | 88 | 88 |
| Synodic 9-8 | 21/34 | 1,156 | 21(34)55 | 54 | 55 | 54.169573 | 54.689759 | 54 | 55 |
| IMO 19 | 34/55 | 1,870 |  | 33 | 34 | 33.486645 | 33.723773 | 33 | 34 |
| Synodic 10-9 | 34/55 | 3,025 | 34(55)89 | 20 | 21 | 20.700835 | 20.860438 | 21 | 21 |
| IMO 210 | 55/89 | 4,895 |  | 13 | 13 | 12.792651 | 12.888208 | 13 | 13 |
| Synodic 11-10 | 55/89 | 7,921 | 55(89)144 | 8 | 8 | 7.9055709 | 7.9663542 | 8 | 8 |
| IMO 311 | 89/144 | 12,816 |  | 5 | 5 | 4.8860820 | 4.9232407 | 5 | 5 |
| Synodic 12-11 | 89/144 | 20,736 | 89(144)233 | 3 | 3 | 3.0198701 | 3.0427860 | 3 | 3 |
| IMO 412 | 144/233 | 33,552 |  | 2 | 2 | 1.8663570 | 1.8805320 | 2 | 2 |
| Synodic 13-12 | 144/233 | 54,289 | 144(233)37 | 77 | 1 | 1.1534570 | 1.1622358 | 1 | 1 |
| IMO 513 | 233/377 | 87,841 |  | 1 | 1 | 0.7128793 | 0.7183005 | 1 | 1 |

Table 2b. The complete framework and the Solar System. Positions, ratios, divisors and Base periods P2, B1, B2.

## Solar System Periods, Pierce Ratios and Divisors below Mercury

Originally the inner region was limited to Synodic 9-8 and Planet 9 (IMO 1) to accommodate Pierce's unused inner reduction ratio of $21 / 55$. Accordingly, relation (2) was applied twice, firstly to the mean periods of Synodic 8-7 and Mercury resulting in 54.689759 days for Synodic $9-8$, and then once again to the latter period and that of Mercury to obtain 33.723773 days for Planet \#9. However, the last two rounded periods are clearly sequential Fibonacci numbers 55 and 34 , an occurrence that allied with the previous sequential pair of periods ( 145 and 88 days versus Fibonacci 144 and 89) provided the impetus to extend the range as far as Planet 13 (IMO 5) in Tables 2a and 2b.
Regarding the present location of Earth near the Mars - Venus synodic position, the calculated synodic period, i.e., Synodic 7-6 = 335 days represents an enigma since it is neither 366-days as required by the divisors, nor it is close to the actual 365.25 days (Julian) and other variants for the year. Although perhaps masked by a possible outward shift by Mars, this still does little to explain the obvious Fibonacci/Phi ratio exhibited by the Venus-Earth periods of revolution expressed in years. In more detail, using modern values for these two adjacent planets the mean periods are $0.61518257: 1$, whereas the reciprocals of Phi and Earth (Unity) are $0.61803398875: 1$. Furthermore, there is also the well-known $5: 8$ ratio between the two planets and associated $5: 8: 13$ Fibonacci resonant triple, i.e., 5 synodic periods of Venus in 8 years with 13 corresponding periods of revolution for this planet. All of which, in addition to the above Fibonacci data from Mercury through IMO5, leads logically enough to the following major expansions.

## The Pierce planetary framework, the Phi-series, and the structure of the Solar System

It is abundantly clear from Table 1b that the final Pierce reduction ratios are successive twinned members of the Fibonacci series, albeit one position removed between the numerators and denominators. Nevertheless, despite the title of Pierce's original publication ${ }^{1}$ and obvious nature of the ratios applied by the latter, the Fibonacci series and related Golden Ratio Phi () :

$$
\begin{equation*}
\operatorname{Phi}()=\sqrt{ }(5 / 4)+1 / 2=1.618033988749895 \tag{3}
\end{equation*}
$$

- are nowhere stressed by Peirce or Agassiz, although this constant clearly plays a major role in the proposed model. This is all the more apparent when it is recalled that the golden section is defined as the division of a line such that the proportion of the smaller section to the larger is identical to the proportion of the larger section to the whole. Whereas the golden ratio can be defined as the limiting value of the ratios of adjacent Fibonacci numbers. It is also clear in the present astronomical context that moving inwards, the limiting value of the inverse alternate Fibonacci ratios applied by Peirce will be ${ }^{-2}$ ( 0.38196601125 ) with reciprocal limit the outward multiplier ${ }^{2}$ (2.61803398875). Furthermore, after the inclusion of the ratios for the intermediate periods between planets the limiting value is ${ }^{-1}$ ( 0.61803398875 ) with a reciprocal limit and a corresponding multiplier of ${ }^{1}$ (1.61803398875), which is Phi itself.

The Phi-series in astronomical context (Periods T, S years, Distance R, Velocity Vi and Vr relative to unity) As it so happens, apart from filling the intermediate gaps introduced by Pierce, relation (1) - the general synodic formula - is already present with one central exception among the four constants just mentioned, i.e., ${ }^{-2,-1,1}$ and . ${ }^{2}$ In short, combined with the calibration and the unification provided by the mean period of Earth ( ${ }^{\circ}=1$ year) the latter become sequential mean periods in years generated by the Phi-series ${ }^{\times}$for successive integer exponents $x=-2,-1,0,1,2$ in the present context. Moreover, with the addition of the next lower integer and also continued outward extensions, integer exponents -3 through 7 generate a complete planetary framework from Mercury to Saturn with all synodic periods included. Beyond this outer region correlation with the solar system parameters begins to diminish, but nevertheless, for the stipulated range the inter-related parameters are as shown in Table 3. Here, following ancient practice it is helpful to include the inverse velocity $V_{i}$ e.g., $V_{i}{ }^{2}=R, V_{i}{ }^{3}=T, V_{i}{ }^{-1}=V r$ (best remembered by the Triple interval $\left[3^{0} 3^{\prime} 3^{2} 3^{3}=1,3,9,27\right]^{7}$ which also pertains to Saturn at perihelion) with the frame of reference (unity) provided by the mean heliocentric distance ( $R$ ) in a.u, mean period of revolution ( $T$ ) in years and mean orbital velocity $(V r)$ of Earth. Thus in the same sense [1, $, 1,1$ ], hence the assignment of the cube to planet Earth and tetradic point-line-area-volume analogy applied to planetary motion. The last three modern periods in days (in red) owe their origins to relation (2) and a 33.0225-day period for IMO 1 by Leverrier (1875). ${ }^{8}$

| PLANETS N Synodics \# | MODERN $T$ (Julian Years) | X | $\begin{gathered} \text { hi-series } T \\ \text { (Years) } \end{gathered}$ | Phi-series (R) Distance (a.u.) | Phi-series (Vi) Inverse Velocity | Phi-series (Vr) <br> Velocity (Ref.1) | MODERN $T$ (Days) |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Saturn 3 | 29.4235194 | 7 | 29.03444185 | 9.446602789 | 3.073532624 | 0.325358512 | 10746.9404 |
| Synodic 4-3 | 19.8588721 | 6 | 17.94427191 | 6.854101966 | 2.618033989 | 0.381966011 | 7253.45303 |
| Jupiter 4 | 11.8565250 | 5 | 11.09016994 | 4.973080251 | 2.230040414 | 0.448422366 | 4330.59576 |
| Synodic 5-4 | 7.84767877 | 4 | 6.854101966 | 3.608281187 | 1.899547627 | 0.526441130 | 2866.36470 |
| M-J Gap 5 | 4.72214968 | 3 | 4.236067977 | 2.618033989 | 1.618033989 | 0.618033989 | 1724.76517 |
| Synodic 6-5 | 3.12552908 | 2 | 2.618033989 | 1.899547627 | 1.378240772 | 0.725562630 | 1141.59949 |
| Mars 6 | 1.88071105 | 1 | 1.618033989 | 1.378240772 | 1.173984997 | 0.851799642 | 686.929711 |
| Earth/Syn 7-6 | 0.91422728 | 0 | 1.000000000 | 1.000000000 | 1.000000000 | 1.000000000 | 365.25(JYR) |
| Venus 7 | 0.61518257 | -1 | 0.618033989 | 0.725562630 | 0.851799642 | 1.173984997 | 224.695433 |
| Synodic 8-7 | 0.39580075 | -2 | 0.381966011 | 0.526441130 | 0.725562630 | 1.378240772 | 144.566223 |
| Mercury 8 | 0.24084445 | -3 | 0.236067978 | 0.381966011 | 0.618033989 | 1.618033989 | 87.9684354 |
| Synodic 9-8 | 0.14474748 | -4 | 0.145898034 | 0.277140264 | 0.526441130 | 1.899547626 | 54.6897591 |
| IMO 19 | 0.09041068 | -5 | 0.076806725 | 0.201082619 | 0.448422366 | 2.230040414 | 33.0225000 |
| Synodic 10-9 | (0.0556507) | -6 | 0.055728090 | 0.145898034 | 0.381966011 | 2.618033989 | 20.3264209 |
| IMO2 10 | (0.0344447) | -7 | 0.040434219 | 0.105858161 | 0.325358512 | 3.073532624 | 12.5818709 |

Table 3. Modern periods $T, S$, Phi-series, exponents ( x ), $T, R$, Velocity Vi (Inverse) and $V r$ (relative to unity).

Returning to the present, notwithstanding the Mars-Jupiter Gap and anomalous location of Earth between Mars and Venus, the Phi-series planetary framework outlined above includes the following properties and relations:

## Heliocentric properties of the Phi-series with respect to unity in the Solar System

1. For any three successive Phi-series periods, the middle period is the product of the periods on either side divided by their difference. Thus, in the same astronomical context, the general synodic formula, relation (1)
2. If two upper adjacent Phi-series periods are known, the third and lower period can be obtained from the product of the two adjacent periods divided by their sum. Thus (in addition to relation 1), synodic relation (2)
3. The underlying constant of the Phi-series planetary model is Phi ( ${ }^{1}=1 / 2 \checkmark 5+1 / 2=1.618033988749895$ ), the limiting value of successive ratios of the Fibonacci series: $1,1,2,3,5,8,13,21,34,55,89,144,233,377,610, \ldots$. (3)
4. For any three successive Phi-series periods, the middle period is the geometric mean of the two periods on either side, as are the means from positions $\pm 2, \pm 3$, etc. Extended geometric means, relations (4), (4E) \& (4F)
5. For every Phi-series period except that of Earth there exists a corresponding Lucas series integer period ( $\quad 3,4,7,11,18,29,47,76,123,199, \ldots$ years) generated by the alternating Phi-Lucas relation: $(T, S)=\left.\right|^{\times} \pm{ }^{-\times} \mid$
Periods of revolution: $T=\left.\right|^{x}-{ }^{-x} \mid$. Intermediate Synodic Periods: $S=\left.\right|^{\times}+{ }^{-x} \mid$ Phi-Lucas relation (5)
6. Pertaining to planet EARTH, the product of the parameters of the planets on either side (Mars and Venus) is UNITY, as are all such Phi-series products, i.e., periods $\pm 2, \pm 3$, etc., both inwards and outwards. Relation (6E)

The limiting Phi-series constants in the present astronomical context are:
A: PLANETS: Mean sidereal periods of revolution, mean heliocentric distances, mean orbital velocities:
Phi-series mean periods of revolution ( $T$ ) decrease ${ }^{-2}$ ( 0.38196601125 ), Inwards (the Pierce limit) (7) Phi-series mean periods of revolution $(T)$ increase ${ }^{2}$ (2.61803398875), Outwards Phi-series mean heliocentric distance $(R)$ increase $\quad 4 / 3$ (1.89954762695), Planets, Outwards Phi-series Planet-to-Planet Velocities (Vr) decrease $\quad^{-2 / 3}$ ( 0.725562630246 ), Planets, Outwards
B: SYNODICS: Mean synodic periods, corresponding heliocentric "distances," mean "orbital" velocities:
Phi-series mean synodic $(S)$ to Planet ( $T$ ) decrease
-1 (0.61803398875), Inwards
Phi-series mean synodic ( $S$ ) to Planet ( $T$ ) increase Phi-series mean synodic $(R)$ to Planet $(R)$ increase Phi-series mean synodic ( $V_{i}$ ) to Planet ( $V_{i}$ ) increase Phi-series mean synodic (Vr) to Planet (Vr) decrease
(1.61803398875), Outwards

2/3 (1.37824077249), Outwards
${ }^{1 / 3}$ (1.17398499671), Outwards

C: GENERATION:
The mean periods of revolution ( $T$ ) and the mean synodic periods ( $S$ ) in years from Mercury to Neptune are generated by the Phi-Series ( ${ }^{\times}$) and integer exponents $x=-3,-2,-1,0,1,2,3,4,5,6,7,8,9,10$ and 11.
(16)

The mean periods of revolution are generated by ODD exponents, mean synodic periods by the EVEN. (17)
D: OVERALL PLANETARY FRAMEWORK with increasing departures beyond Saturn (periods $T, S$ in years). Period divisors, modern values, exponents, Lucas series and Phi-series framework are shown in Table 3s. n.b., the Phi-series also includes each key Pheidian constant as a mean period ( $T, S$ ), a mean distance $(R)$ and both velocities ( $V$ r \& Vi) with the latter ( 0.381966011 ) at Synodic 10-9) not shown.
(18)

| $\frac{\text { DIV. }}{(\mathrm{syn})}$ | PLANETS N Synodics \# | MODERN T,S (Julian years) | exp. | LUCAS (years) | $T, S$ | Phi-series ( $R$ ) <br> Distance (a.u.) | Phi-series (Vr) <br> Velocity (ref.1) | Phi-series (Vi) <br> Velocity (Inv.) |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | Neptune 1 | 163.7232045 | 11 | 199 | 99.0050294 | 912 | . 171282103 | 321602 |
| (1) | Synodic 2-1 | 171.4442895 | 10 | 123 | 122.9918694 | 4.73152718 | 082619 | 4.973080251 |
| $\underline{2}$ | Uranus 2 | 83.7474068 | 9 | 76 | 76.01315562 | 17.94427191 | 0.236067977 | 4.236067978 |
| (4) | Synodic 3-2 | 45.3598213 | 8 | 47 | 46.97871376 | 13.01969312 | 0.277140264 | 3.608281187 |
| $\underline{6}$ | Saturn 3 | 29.4235194 | 7 | 29 | 29.03444185 | 9.446602789 | 0.325358512 | 3.073532624 |
| (9) | Synodic 4-3 | 9.8588721 | 6 | 18 | 7.94427191 | . 854101966 | . 3819660 | . 61803398 |
| 15 | Jupiter 4 | 11.8565250 | 5 | 11 | 11.09016994 | 4.973080251 | 0.448422366 | 2.23004041 |
| (25) | Synodic 5-4 | 7.84767877 | 4 | 7 | 6.854101966 | 3.608281187 | 0.526441130 | . 89954762 |
| 40 | M-J Gap 5 | 4.72214968 | 3 | 4 | 4.236067977 | 2.618033989 | 0.618033989 | 1.61803398 |
| (64) | Synodic 6-5 | 3.12552908 | 2 | 3 | 2.618033989 | . 899547627 | 0.725562630 | 378240772 |
| 104 | Mars 6 | 05 | 1 | (2) | 1.618033989 | 1.378240772 | 0.851799642 | 1.17398499 |
| (169) | Earth/Syn 7-6 | 0.91422728 | 0 | 1 | 1.000000000 | 1.000000000 | . 000000000 | . 000000000 |
| 273 | Venus | 0.61518257 | -1 |  | 0.618033989 | 0.725562630 | 1.173984997 | 0.851799642 |
| (441) | Synodic 8-7 | 0.39580075 | -2 | - | 0.381966011 | 0.526441130 | 1.378240772 | 0.725562630 |
| 714 | Mercury 8 | . 240 | -3 | (Rel. 5) | 0.236067978 | 0.381966011 | 1.6180339 | 0.61803 |

Table 3s. Divisors, modern periods (T\&S), Phi-series (x) T, S, Lucas T, S, Phi-series R, Velocity Vr \& Inverse Vi.

Disparities in the modern Solar System from Mercury through Saturn with emphasis on the Pierce Divisors and insights from the above are shown in Figure 1.


Fig. 1. The Pierce planetary framework, Solar System Mercury-Saturn, Mars-Jupiter Gap and the location of Earth.

## Similarities and Disparities

Applied to the present Solar System, the Phi-series from $x=-3$ to 7 yields a planetary framework which includes all the intermediate (synodic) periods from Mercury through Saturn plus periods for theoretical planet \#5 and both adjacent intermediate synodic intervals. Beyond Saturn correlation diminishes with distance, while the ratios of the of the integral Lucas series increasingly approach Phi itself. Whereas, moving inwards, ratios of the period divisors also begin to approach the same fundamental constant. Nevertheless, two identical disparities in the Solar System are indicated by all three sequences: (1) the absence of a planet between Jupiter and Mars, and (2), the unexpected presence of a planet between adjacent Mars and Venus, namely Earth. Moreover, in addition to this location there is also a marked difference between the calculated intermediate period for Earth of 335 days and the 365-day year.

In so much as Venus and Earth have the lowest eccentricities among the planets and their periods of revolution are also closest to their Phi-series equivalents - with zero error for Earth - the position of the latter can be examined in terms of residual effects of the Phi-series with relation 6E a possible factor. This, however, is difficult to investigate because of the missing periods between Mars and Jupiter, and also the accepted absence of planets below Mercury.

About the only option remaining pertains to the periods of Jupiter and IMO 1, i.e., the periods corresponding to Phi-series exponents +5 and -5 which yield a product of exactly 1 year. Whereas in the Solar System the mean period of Jupiter of 11.85652502 Julian years and that of IMO 1 ( 0.09035592 years ) yields a product of 1.071307238, with the replacement of IMO 1 by the mean synodic month resulting in 0.9586044 years. Unity does, however result from a period of 30.8058220 days from the reciprocal of Jupiter's mean period, a concept which owes its origins to Friberg's approach to AO 6484, a Babylonian mathematical text concerned with the number 0;59,15,33,20 and its reciprocal $1 ; 00,45 .{ }^{9}$ The product is necessarily unity with a sum of $2 ; 0,0,33,20$ and $1 ; 0,0,16,40$ for the half. ${ }^{9}$ Which, albeit radical shifts in both time and place, can be considered in terms of elliptical parameters for the orbit of Earth. This is an unexpected bi-product of a reappraisal of the 1964 analyses by A. Aaboe ${ }^{10}$ of a possible daily increment of $0 ; 0,1,32,42,13,20^{\circ}(0.000480109739369)$ for the velocity of the "Sun" in BM 37089, a Babylonian lunar fragment.

The relevance of the latter is that the value $0 ; 59,15,33,20^{\circ / \mathrm{Day}}$ can be shown to be inherent in data in Aaboe's study which corresponds to a period of exactly 364.5 years. This value is shown below in the last column of Table 1A from an expansion of Aaboe's analyses incorporating a Babylonian System B varying velocity function for planet Earth:
"... Although Aaboe surmises that the original table may have supplied daily longitudes for a complete year, he gives a partial restoration since neither the maximum nor the minimum values are present. It is, however, sufficient to give Aaboe's daily longitudes and differences for lines 5 through - 5 plus added corresponding lengths of the year in days to show that line 0 is the closest to the Sidereal year:

| Line \# | ${\text { Col. II }\left(\text { Longitude }^{\circ}\right)}^{c}$ | $\Delta$ Col. II (Daily velocity ${ }^{\circ}$ ) | $T$ (years, added) |
| :--- | :--- | :--- | :--- |
| Line -5 | $[8 ; 51,51,51,6,40]$ | $0 ; 59,17,17,2,13,20$ | 364.32289859 |
| Line -4 | $[9 ; 51,6,40]$ | $\underline{0 ; 59,15,33,20}$ | 364.5 |
| Line-3 | $[10 ; 50,20,29,37,46,40]$ | $0 ; 59,13,49,37,46,40$ | 364.67727367 |
| Line-2 | $[11 ; 49,32,35,33,20]$ | $0 ; 59,12,5,55,33,20$ | 364.85471987 |
| Line -1 | $[12 ; 48,42,57,46,40]$ | $0 ; 59,10,22,13,20$ | 365.03233883 |
| Line 0 | $[13 ; 47,51,36,17,46,40]$ | $\underline{0 ; 59,8,38,31,6,40}$ | 365.21013081 |
| Line 1 | $[14] ; 46,58,[31,6,40]$ | $0 ; 59,6,54,48,53,20$ | 365.38809607 |
| Line 2 | $15 ; 46,[3,42,13,20]$ | $0 ; 59,5,11,6,40$ | 365.56623485 |
| Line 3 | $16 ; 45,7,[9,37,46,40]$ | $0 ; 59,3,27,24,26,40$ | 365.74454742 |
| Line 4 | $17 ; 44,8,[53,20]$ | $0 ; 59,1,43,42,13,20$ | 365.92303402 |
| Line 5 | $18 ; 43,[8,53,20]$ | $0 ; 59$. | 366.10169492 |

Table 1A. Daily solar positions and velocities with periods $T$ added to Aaboe (1964:32).
This demonstrates that from a modern perspective the mean daily velocity from line 0 of $0 ; 59,8,38,31,6,40^{\circ / d}$ and the 365.21013081 -day year are optimum for $(u)$ and $(T)$ respectively. But not quite. In order to restore the longitudes and velocities for the entire table, the period $T$ turns out to be exactly 364 days. Thereafter, with (d) given, (u) from Line 0 , and $T=364$ days, relation $(X)$ is reduced to: $(M, m)=0 ; 59,8,38,31,6,40 \pm 0 ; 2,37,17,2,13,20$ which produces the following six-sexagesimal place values for the apsidal velocities and the daily velocities in between.

$$
\begin{aligned}
& \text { Minimum daily velocity }(m)=0 ; 56,31,21,28,53,20^{\circ / \text { day }} \\
&\text { (abbrev. } 0 ; 56,30) \\
& \text { Mean daily velocity }(u)=0 ; 59,8,38,31,6,40^{\circ} / \mathrm{day} \\
& \text { Maximum daily velocity }(M)\text { (abbrev. } 0 ; 59,9) \\
& \text { Mat1,45,55,33,20,00/day }\text { (abbrev. } 1 ; 1,46) \\
& \text { eccentricity }(e)=0.0295589 .
\end{aligned}
$$

The occurrence of $0 ; 59^{\circ}$ in line 5 of column 3 suggests choice rather than coincidence and there are other matters of interest in addition." [Excerpt from "Aaboe64 Revisited"].

The above dialogue concludes with an associated ellipse and additional variants which are beyond the scope of the present study. Except to note that Friberg's analysis mentioned earlier is accompanied by a two-part figure for the Babylonian mathematical procedure known as "Completing the Square." The latter, however, in consort with the calculation of the heliocentric distances $R$ (by a procedure provisionally named here "Completing the Cube" inherent in Old Babylonian mathematical text VAT 8547) suggests that these procedures ultimately concern the derivation of the parameters of ellipses for Earth and the major superior Planets. In so much as the eccentricities (e) are small (e.g., that of Earth is 0.01670862 ) the orbits appear to be almost circular, which provides an impetus to revisit Babylonian mathematical texts with accompanying "circles" and non-integer numerical values close to unity or 2 . Thus possible semi-major (a) and major axes ( $2 a$ ) for Earth/Sun ellipses, e.g., although conceivably with alternate meanings:
"Fig. 3. 1. 12. MS 3050. An OB round hand tablet with square inscribed in a circle." Friberg (2005:135). ${ }^{.1}$
"Fig. 16. 7. 3. UET/67 2222 rev. A square side algorithm using elimination of square factors." Friberg (2006:401) ${ }^{12}$
"Fig. 16. 7. 4. 1 st. Si. 428. Computation of the square side $2 ; 02,02,02,05,05,04$." Friberg (2006:403). ${ }^{13}$
where the first figure appears to be a rough rectangle with diagonals inscribed in an equally rough ellipse.
Seeking further enlightenment the inquiry leads to Babylonian planetary and luni-solar parameters, but before this it is necessary to caution the casual reader about prevailing nihilistic views concerning Babylonian astronomy, especially ill-founded claims that the Babylonians had neither a fictive approach to orbital motion nor any planetary model whatsoever. Long overdue additional research shows that nothing could be further from the truth.

But before proceeding, the notation, conventions and additional data in this context are introduced for those who may be unfamiliar with this relatively obscure material, along with standard definitions of astronomical terms, and in particular, luni-solar and planetary parameters in both modern and Babylonian contexts.

## Sexagesimal notation, Units, Time, and Motion

Sexagesimal numbers 1 to 59 are separated by commas with equivalent decimal place locations indicated by semicolons, thus in addition to hours; minutes and seconds, the thirds, fourths, fifths, sixths, sevenths, etc. For example, the Old Babylonian estimate for the square root of 2 rounded at the third place is $1 ; 24,51,10^{13}$ with the exact value for the Babylonian mean synodic arc of Saturn ${ }^{13} 12 ; 39,22,30^{\circ}\left(12.65625^{\circ}\right)$ with a corresponding mean synodic time of $1 ; 2,6,33,45$ years ( 1.03515625 versus the modern mean synodic period for this planet of $1.035182135 \ldots$ years). Days, degrees, months and "tithis" (thirtieths) are denoted by the superscripts $n^{d}, n^{0}, n^{m}, n^{r}$ with the predominant Babylonian mean synodic month (MSM) of $29 ; 31,50,8,20^{d}\left(29.5305941358 . . .{ }^{d}\right)$ represented by superscript ${ }^{M}$.

Next, expanded later, definitions and tools for the present study include the following luni-solar constants:
(1) DAY: Daily axial rotation and daily sidereal motion of Earth with subdivisions of the 24 -hour day for time \& angular motion which far exceed modern usage, extending from $360^{\circ}$ per day through Large Hours ( $30^{\circ}$ ), Hours ( $15^{\circ}$ ), Minutes and Seconds, etc., down to 50 seconds of arc $\left(0 ; 00,50^{\circ}\right)$.
(2) MONTH: MEAN SYNODIC MONTH of 29;31,50,8,20 days $=29.5305941358^{d}$ with last base-60 pair rounded for convenience. Even so it is still quite accurate; the modern estimate is 29;31,50,7,30 days.
(3) YEAR: SIDEREAL YEAR of $12 ; 22,8$ Mean synodic months $=365 ; 15,38,17,44,26,40$ days ( 365.2606376886 ). Although the latter is high compared to the modern estimate of $365.2564^{d}$ it is almost certainly selected for convenience. A better estimate for the sidereal year is also available from the accurate Babylonian mean sidereal month of $27 ; 19,18^{\text {d }}$ and above mean synodic month which generate a year of 365.2564698 days.
(4) METHODOLOGY: Explanations of the fundamental motions involved according to the methods laid out in the Babylonian procedure texts and related data determined from the Babylonian end products, i.e., the Ephemerides. And in addition, the implications of the Earth/Sun duality in the Babylonian context.
(5) Closely associated to (4), the underlying formulas required to assess Babylonian results and procedures. In this case, since Babylonian planetary theory deals to a considerable extent with synodic motion, and the latter understanding is also applicable to the lunar component, the computation of synodic cycles, synodic periods and synodic arcs also play a role in the current investigation, the following especially:

## Synodic periods and synodic formulas

The synodic period $(S)$ or lap-cycle between two Solar System planets with mean periods of revolution $T_{1}$ and $T_{2}$ is given by the general synodic formula for co-orbital bodies applied earlier to the Pierce data :

$$
\begin{equation*}
\text { Synodic period } S=\frac{T_{1} \cdot T_{2}}{T_{1}-T_{2}}\left(T_{1}>T_{2}\right) \tag{1}
\end{equation*}
$$

along with the simplified standard synodic formulas for the Superior and Inferior planets:

$$
\begin{equation*}
\text { Superior planets, } S_{\mathrm{s}}=\frac{T_{\mathrm{s}}}{T_{\mathrm{s}}-1} \quad \text { (1s) } \quad \text { Inferior planets, } S_{\mathrm{i}}=\frac{T_{\mathrm{i}}}{1-T_{\mathrm{i}}} \tag{1i}
\end{equation*}
$$

augmented, if required, by synodic relation (2) where periods $T_{1}$ and $S$ are known and period $T_{2}$ is of interest:

$$
\begin{equation*}
\text { Period } T_{2}=\frac{S \cdot T_{1}}{S+T_{1}}\left(S>T_{2}\right) \tag{2}
\end{equation*}
$$

Synodic relations (1), (2) and modern equivalents all have roles to play in what follows, but relation (1) in full has a further application arising from the inclusion of the mean synodic month in Babylonian planetary theory beyond calendaric considerations. Although obvious, this was either missed or - for whatever reasons - ignored by noted authority Otto Neugebauer. More on this matter later.

As for the relevance of Babylonian astronomy in the presence context, further examination the mathematical cuneiform texts from the Old Babylonian Period (1900 BCE - 1650 BCE), ${ }^{14}$ the Babylonian astronomical diaries from $652-62$ BCE, ${ }^{15}$ details in the Babylonian astronomical "procedure" texts and the resulting Ephemerides of the Seleucid Era ( $310 \mathrm{BCE}-75 \mathrm{CE})^{16}$ represent an extensive source of largely misrepresented and/or misunderstood information. Included here are specific parameters with descriptions in the procedure texts concerning their determination, sufficient details, in fact, for the heliocentric concept and refined laws of planetary motion to be added to the already complex mathematics of the Old Babylonian Period. The acceptance of which is adversely influenced by the time line between the sources and lack of connectivity with the earliest in terms of known astronomical concepts.
OLD BABYLONIAN PERIOD (1900 BCE - 1650 BCE). Advanced Mathematical Cuneiform Texts.

Text-Fig 1. Old Babylonian Mathematics, Babylonian Astronomical Diaries and Seleucid Era Astronomy.
PRIMARY WORKS, REFERENCES \& JOURNALS (Centaurus, ISIS, JCS, JHA, JNES, JRASC, Nature, Sumer)
Essay on Classification. Louis Agassiz (1852).
On the Relation of Phyllotaxis to Mechanical Laws. Arthur H. Church (1904).
Harmonic Proportion and Form. Samuel Colman (1912).
The Curves of Life. Sir Theodore Andrea Cook (1914).
Dialogues Concerning Two New Sciences. Galilei Galileo (1608), (1914)
Sternkunde und Sterndienst in Bable. Franz X. Kugler (1914), (1924).
On Growth and Form. Sir d'Arcy Wentworth Thompson (1917).
Dynamic Symmetry. Jay Hambridge (1920).
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The Babylonian Theory of the Planets. Noel Sverdlow (1998).
The Golden Ratio. The Story of Phi. Mario Livio (2002).
Unexpected links between Egyptian and Babylonian Mathematics. Jöran Friberg (2005).
Amazing traces of a Babylonian origin in Greek Mathematics. Jöran Friberg (2006).
A remarkable collection of Babylonian Mathematical Texts. Jöran Friberg (2007).
Mathematics in Ancient Iraq : A Social History. Eleanor Robson (2008).

## Introduction

Although the roles of synodic relations (1) and (2) with respect to the Solar System periods and synodic cycles are not entirely surprising the two relations are nevertheless both underlying elements of the structure of the Phi-series planetary framework. Just how well the Babylonian luni-solar material reflects this is another matter, but assuredly the subject is worthy of further investigation, especially with respect to attested Babylonian periods and velocities for the five planets known in Antiquity. But then again, the Fibonacci, Lucas and the Phi-series are all considered to be relatively recent in both origins and understanding, hence the following introduction to the historical side of the matter.

## I. The Fibonacci and the Lucas series in early times.

Although the first of these two elementary series is still credited to Fibonacci (ca.1175-1240 CE) and likewise the second to Francois Lucas (1842-1891), as Thompson pointed out over a century ago, ${ }^{17}$ it is unlikely that the former would have escaped the attention of Greek philosophers or even earlier inquiring minds. Furthermore, this same argument applies equally (if not more so) to the latter series ( $1,3,4,7,11,18,29,47$ ) since it is simply the next additive sequence after the Fibonacci, i.e., $1,1,2,3,5, \ldots$ is followed by: $1,3,4,7, \ldots$... (the Lucas), then: $1,4,5,9, .$. and $1,5,6,11, \ldots$ etc., all with the same limiting ratio () between adjacent pairs. The last mentioned (provisionally the Penta series 1,5, 6,11, $17,28, .$. ) also includes the first two perfect numbers 6 and 28 (numbers equal to the sum of their own parts). And eventually, the convenient approximation for the Golden ratio of $809 / 500=1618 / 1000$, thus $1.618(1 ; 37,4,48)$.

## II. Babylonian Jupiter/Saturn mean synodic arcs; the Phi-series and the Golden ratio

Both historically and in astronomical terms, the ratio $5: 6$ is known to play an underlying role in the location of the extremal synodic arcs for Jupiter ${ }^{18}$ and Saturn ${ }^{19}$ in the Babylonian astronomical cuneiform texts of the Seleucid Era (310 BCE-75 CE). Furthermore, despite current dismissive views on this subject, another point of relevance is found in the Babylonian estimates for the sidereal periods of revolution for Jupiter (11;51,40 $=11.86111^{*}$ years) and Saturn $\left(29 ; 26,40=29.444^{*}\right.$ years) which provide the basis for the mean synodic arcs ( $u$ ) according to Babylonian System B. In particular, it is the ratios of these synodic arcs - $33 ; 8,45^{\circ}\left(33.1458333^{*}\right)$ for Jupiter ${ }^{20}$ and $12 ; 39,22,30^{\circ}(12.65625)$ for Saturn ${ }^{21}$ - which are of immediate interest, since:

$$
\begin{aligned}
& \frac{\text { Saturn }(u)}{\text { Jupiter }(u)}=\frac{12.65625}{33.14583333^{*}}=0.38183534 \text { versus }^{-2}=0.38196601125 \text {, the Pierce Limit, Phi-series relation } \\
& \frac{\text { Jupiter }(u)}{\text { Saturn }(u)}=\frac{33.14583333^{*}}{12.65625}=2.61893004 \text { versus }^{2}=2.61803398875 \text {, Planet-to-Planet Phi-series relation }
\end{aligned}
$$

whereas the difference between the two mean synodic arcs, i.e., Jupiter (u) -Saturn (u)=20;29,22,30́ (20.48958333*) not only provides the arc for the difference cycle SD1 between the two planets (Synodic 4-3 in the Peirce framework) but also two further inter-related ratios of similar interest:

$$
\begin{aligned}
& \frac{\text { Jupiter }(u)}{S D 1(u)}=\frac{33.14583333^{*}}{20.48958333^{*}}=1.61769192 \text { versus }=1.61803398875, \text { Planet-to-Synodic Phi-series relation } \quad(12)_{J / D} \\
& \frac{S D 1(u)}{\text { Saturn }(u)}=\frac{20.48968333^{*}}{12.65625}=1.61893004 \text { versus }=1.61803398875 \text {, Synodic-to-Planet Phi-series relation } \quad(12)_{D / S}
\end{aligned}
$$

## III. Babylonian Jupiter/Saturn mean synodic arcs and the Fibonacci series

In addition to this pair of mean synodic arcs, System $A^{\prime}$ for Jupiter ${ }^{22}$ features an intermediate arc of $33 ; 45^{\circ}\left(33.75^{\circ}=u_{2}\right)$ as opposed to ( $u$ ), the mean synodic arc of $33 ; 8,45 .^{\circ}$ Retaining Saturn's mean synodic arc of $12 ; 39,22,30^{\circ}$ but using $33.75^{\circ}$ for Jupiter and new difference arc $S D 1^{\prime}=21 ; 5,37,30^{\circ}(21.09375)$ the divisions for the new arcs now yield the following familiar Fibonacci ratios which suggests the previous relations are unlikely to be coincidental or unknown;

$$
\begin{aligned}
& \frac{\text { Jupiter }\left(u_{2}\right)}{\text { SD1 }(u)^{\prime}}=\frac{33.75}{21.09375}=1.6(1 ; 36) . \text { Fibonacci ratio }(8 / 5) \\
& \frac{S D 1(u)^{\prime}}{\text { Saturn }(u)^{\prime}}=\frac{21.09375}{12.65625}=1.666^{*}(1 ; 40) . \text { Fibonacci ratio }(5 / 3) \\
& \frac{\text { Jupiter }\left(u_{2}\right)}{\text { Saturn }(u)}=\frac{33.75}{12.65625}=2.666^{*}(2 ; 40) . \text { Fibonacci ratio }(8 / 3)
\end{aligned}
$$

## IV. Jupiter and Saturn mean value ratios for Babylonian Systems A and B

Remaining with Jupiter and Saturn, there is a major difference between the two primary methods for dealing with varying synodic motion (Systems A and B) with the two-velocity configurations of System A using a minimum arc ( $w$ ) and a maximum arc ( $W$ ) sensibly understood to be apsidal velocities with pheidian elements in an associated 5:6 ratio. For Jupiter the minimum and maximum synodic velocities (or apsidal arcs) are $30^{\circ}$ and $36^{\circ},{ }^{23}$ whereas for Saturn the minimum ( $w$ ) and the maximum ( $W$ ) have a marked difference in the number of sexagesimal places, i.e., $(w)=11 ; 43,7,30^{\circ}(11.71875)$, and $\left.(W)=14 ; 3,45^{\circ}(14.0625)\right)^{24}$

On further examination, however, it seems possible that the latter set may have originated from the former since the seemingly more accurate apsidal synodic arcs for Saturn can be derived from the Jupiter data by simple division, i.e., $36^{\circ} / 2.56=14 ; 3,45^{\circ}(14.0625)$ and $30^{\circ} / 2.56=11 ; 43,7,30^{\circ}(11.71875)$. Plus one further point; the common divisor is also the square of Fibonacci ratio $8 / 5\left(1.6^{2}=2.56\right)$, thus a practical reduction factor for the periods of revolution of these two adjacent major planets in keeping with the Fifths and the Sixths of the final Pierce framework.

It is, however, more complicated than this, for even though Jupiter's new value from System $\mathrm{A}^{\prime}\left(u_{2}=33 ; 45^{\circ}\right.$ as used in relations ( 12$)_{/ / D F}$ though ( 8$)_{/ J S F}$ ) resulted in three Fibonacci ratios using this constant, it is not the actual mean value for Jupiter, which is $1 / 2(W+w)=\left(u_{3}\right)=33^{\circ}$. Whereas the mean value from Saturn's System A is in turn found to be $1 / 2\left(14 ; 3,45^{\circ}+11 ; 43,7,30^{\circ}\right)=\left(u_{4}\right)=12 ; 53,26,5^{\circ}(12.890625)$ with the ratio between the two new mean values now:

$$
\begin{equation*}
\frac{\text { Jupiter }\left(u_{3}\right)}{\text { Saturn }\left(u_{4}\right)}=\frac{33}{12.890625}=2.56=1.6^{2} . \text { Fibonacci ratio }(8 / 5)^{2} \tag{8}
\end{equation*}
$$

while the ratio between Jupiter $\left(u_{2}\right)$ and Saturn $\left(u_{4}\right)$ is:

$$
\begin{equation*}
\frac{\text { Jupiter }\left(u_{2}\right)}{\text { Saturn }\left(u_{4}\right)}=\frac{33.75}{12.890625}=2.6181818^{*}=\text { Fibonacci ratio } 144 / 55 \tag{8}
\end{equation*}
$$

with a reciprocal of: $\quad \frac{\text { Saturn }\left(u_{4}\right)}{\operatorname{Jupiter}\left(u_{2}\right)}=\frac{12.890625}{33.75}=0.3819444^{*}=$ Fibonacci ratio 55/144
and a corresponding ideal growth angle ( $360^{\circ} \cdot 0.3819444^{*}$ ) of $137.5^{\circ}$.
Lastly, with a new difference arc SD1 of $(33.75-12.890625)=20.859375(u)$ ", relation $(12)_{/ / D F}$ now becomes:

$$
\begin{equation*}
\frac{\text { Jupiter }\left(u_{2}\right)}{\text { SD1 }(u)^{\prime \prime}}=\frac{33.75}{20.859375}=1.6179775281=\text { Fibonacci ratio }(144 / 89) \tag{12}
\end{equation*}
$$

At which point Babylonian astronomy in general and the origins of these mean synodic arcs in particular begin to assume an unexpected level of importance despite almost universal dismissal of Babylonian methodology at the present time. For this reason the Babylonian observational reference frames and resulting luni-solar parameters in particular offer a minor introduction to the optional excursus at the end of Part 1.

## V. Babylonian luni-solar parameters and Phi-series/synodic relations (1) and (2)

The inclusion of luni-solar parameters in the present context gives rise to the following added abbreviations, names, descriptions and periods in base-60 with decimal equivalents. All bar the tropical month and the tropical year were gleaned from leading authority O . Neugebauer's barely readable sexagesimal analyses rendered less understandable by the latter's non-model approach to Babylonian planetary theory. Because of these problems the following tables are largely prior analytics initially limited to mean values for synodic relation (1) subroutines applied in Table AP2

| Abbr. | Astronomical Names and Standard Descriptions | Babylonian periods | Decimal days |
| :--- | :--- | :--- | :--- |
| MSM: | Mean Synodic month (new moon to new moon). | $29 ; 31,50,8,20$ (rounded) | $29.530594136^{\text {d }}$ |
| MSID: | Mean Sidereal month (fixed star to fixed star). | $27 ; 19,18$ (rounded) | $27.321666667^{\text {d }}$ |
| MTROP: | Tropical month (equinox to equinox; text, calc., added). | $27 ; 19,17,45$ (rounded) | $27.321574074^{\text {d }}$ |
| MAN: | Anomalistic month (perigee to perigee). | $27 ; 33,20$ (unrounded) | $27.5555555555^{\text {d }}$ |
| MDRA: | Draconic month (node to node). ACT. | $27 ; 12,44$ (rounded) | $27.212222222^{\text {d }}$ |
| MDRA2: | Draconic month (node to node), ACT. calc. | $27 ; 12,43,59,40$ (rounded) | $27.212220679^{\text {d }}$ |
| SYR: | Sidereal year (fixed star to fixed star). calc. 12;22,8•MSM | $365 ; 15,38,17,44,26,40$. | $365.26063769^{\text {d }}$ |
| SYR2 | Sidereal year (fixed star to fixed star). calc. (MSM : MSID). | $365 ; 15,23,17,30$. | $365.25646991^{d}$ |
| TYRB: | Tropical year (equinox to equinox; (Bab. ACT 210, Sect.3) | $365 ; 14,4,51$ | $365.24579167^{\text {d }}$ |
| AYR: | Anomalous year (perihelion to perihelion (calc., added). | $365 ; 15,34,18,22,58,51$, | $365.25952955^{\text {d }}$ |
| EYC: | Eclipse cycle (lunar node to lunar node) (text/mult/calc.). | (5458/465)•MSM. | $346.61931784^{\text {d }}$ |
| AYC: | Anomalistic cycle (text/mult/calc; added) | (251/18)•MSM. | $411.78772933^{\text {d }}$ |

Table AP1. Astronomical terms, Babylonian mean luni-solar periods and decimal equivalents I

Next, the role played by the two primary Phi-series/synodic relations in the present context should also be noted:

$$
\begin{equation*}
\text { Synodic period } S_{2}=\frac{T_{1} \cdot T_{3}}{T_{1}-T_{3}}\left(T_{1}>S_{2}>T_{3}\right) \quad \text { (1) } \quad \text { Inner period } T_{3}=\frac{T_{1} \cdot T_{2}}{T_{1}+T_{2}}\left(T_{1}>T_{2}>T_{3}\right) \tag{2}
\end{equation*}
$$

Significantly, the mean synodic month (MSM $=T_{1}$ ) and the tropical month (MTROP $=T_{1}$ and $T_{2}$ ) also play a role in the comparisons between other Babylonian luni-solar cycles, mean luni-solar periods and the modern values with variants of Phi-series/synodic relation (1) predominating in Table AP2.

| \# | Cycles and/or Periods | Subroutine ( $\left.T_{1}>T_{2}>T_{3}\right)$ | Mean periods | Modern equivalents/; (sources) Relation | Relations (1x) |
| :---: | :---: | :---: | :---: | :---: | :---: |
| [1] | Eclipse cycle (EYC) | (MSM : MDRA) | 346.619576 days | (Modern: 346.620107 days; (calc.) | (1e) |
| [2] | Anomalistic cycle (AYC) | (MSM : MAN) | 411.780405 days | (Modern: 411.783870 days; (calc.) | (1a) |
| [3] | Nodal cycle (days) | $(\mathrm{MTROP}: \mathrm{MDRA})=$ | 6797.54400 days | (Modern: 6798.26051 days; Tables: 6798) | s: 6798) (1n) |
| -- | Nodal cycle (years) | $(\mathrm{MTROP}: \mathrm{MDRA})=$ | 18.6108756 years | (Modern: 18.6128373 years:(calc.,) | ) (1n) |
| [4] | Lunar perigee | (MAN : MTROP) | 3231.88186 days | (Modern: 3231.56072 days; Tables: 3232) | s: 3232) (1p) |
| [5] | Sidereal year (SYR. MYR) | (MSM •12;22,8) | 365.260637 days | (Expressed in mean synodic months = MYR) | hs = MYR) -- |
| [6] | Sidereal year (SYR2,calc.) | (MSM : MSID) | 365.256469 days | (Modern: 365.256365 days; Tables) | s) (1s) |
| [7] | Tropical year (TYR, calc.) | (MSM : MTROP) $=$ | 365.244059 days | (Modern: 365.242189 days; Tables) | s) (1t) |
| [8] | Tropical year (TYR, text) | (TYRB: 18-yr pd $=$ | 365.245792 days | 365;14,44,51 days (ACT 210, Sec. 3) | ) |
| [9] | Anomalistic year (AYR) | MSM• $360^{\circ} /\left(u^{\circ}\right)=$ | 365.259529 days | (Modern: 365.259641 days; Tables) $u=29 ; 6$ | s) $u=29 ; 6,19,20^{\circ}$ |
| [10] | SAROS, 19 EYC or | (223 MSM, calc.) = | 6585.32249 days | (Modern: 6585.32163 days; Tables) | ) -- |

Table AP2. Astronomical terms, Babylonian luni-solar cycles and decimal equivalents II.
For example, although the slightly too large yet practical sidereal year SYR (\#[5]) is 12;22,8 mean synodic months or 365.260637 days, the more accurate value (SYR $2=365.256469$ days) is readily available by way of synodic relation (1) utilizing the Babylonian mean synodic month (MSM $=29 ; 31,50,8,20$ days) and the mean sidereal month (MSID = 27;19,18 days):

$$
\text { SYR2 }=\frac{\text { MSM } \cdot \text { MSID }}{\text { MSM + MSID }}=365.256469811878(365 ; 15,23,17,28,45,43 \text { days })
$$

## VI. The Tropical month from Babylonian luni-solar parameters

Also noteworthy are the extended Babylonian luni-solar cycles, especially those stated in eight lines of lunar text No. ACT 210, Section $3 .{ }^{23}$ Although rarely recognized as such, they include one of the more contentious issues likely to arise in this context, i.e., presence of the Tropical month and the Tropical year in Babylonian astronomy. The latter ([8] in Table AP2) occurs as " $1,49,45,19,20$ days of 18 years of the moon," ${ }^{25}$ yielding $354 ; 14,44,51$ days, which is superior to that used by Claudius Ptolemy ( $365 ; 14,48=365.24666^{*}$ days). More helpful, however, the presence of a tropical year supplies the means for determining a theoretical length for the Tropical month in Babylonian astronomy.

Applying a value for the Tropical year (TYRB) of 365;14,44,51 days and mean synodic month (MSM) of 29;31,50,8,20 days, an estimate for the tropical month (MTROP) is available from synodic relation (2) i.e., subroutine TYRB : MSM:

$$
\begin{equation*}
\text { MTROP }=\frac{\text { TYRB } \cdot M S M}{\text { TYRB }+ \text { MSM }}=27.32160692(27 ; 19,17,47,5, \ldots \text { days }) \tag{2tr}
\end{equation*}
$$

which rounds conveniently to $27 ; 19,17,45$ days and the even more convenient Babylonian estimate of 29;19,17,40 days for (perhaps) ACT 210 Section 3. The assignment of $365 ; 14,44,51^{\text {d }}$ for a Babylonian tropical year was previously proposed by Hartner in an erudite discussion concerning the tropical year and precession which ended as follows ${ }^{26}$

> The inevitable conclusion to be drawn from the preceding demonstrations is, that in Babylonia under Achaernenian rule at the latest in 503 B.C., a clear distinction is made between the length of the tropical year: $A=365 ; 14,48,33,37^{\text {d }}$ (possibly already then found exchangeable in practice with $\mathrm{Ar}=365 ; 14,44.51^{d}$ ) and that of the sidereal year as underlying System B: PB ' $=365 ; 15: 34,18,1 \ldots{ }^{\text {d }}$ (italics supplied)
> Willi Hartner, "The Young Avestan and Babylonian Calendars and the Antecedents of Precession." JHA, X,1979:1-22.

## VII. Precession and the Babylonian Sidereal/Tropical years

Thus once again Phi-series/General synodic relation (2) is indicated, albeit with respect to mean values, whereas although the standard sidereal year [5] and tropical year [8] are both on the high side, their difference nevertheless yields a Seleucid Era value (perhaps known, perhaps not) for annual precession of 0;0,52,40,41,... ${ }^{\circ}$ and 24,602 years for the full cycle.

## VIII The Anomalistic year

Unlike the derivations based on synodic relations the anomalistic year can be obtained from the mean sidereal arc of Earth ( $29 ; 6,19,20^{\circ}$ ) per mean synodic month of $29 ; 31,50,8,20^{d}$. This ratio yields a daily velocity of $0 ; 59,8,9,43,22,7, \ldots{ }^{\circ}$

$$
\begin{equation*}
\text { Mean daily velocity of Earth }=\frac{29 ; 6,19,20^{\circ}}{29 ; 31,50,8,20^{d}}=0 ; 59,8,9,43,22,7, \quad\left(0 ; 59,8,9,43,20 \text { rounded }=u^{\prime}\right) \tag{3m}
\end{equation*}
$$

for a corresponding year of $365 ; 15,34,18,22,58,51,40^{\mathrm{d}}$ or 365.2595295 ... days. The modern estimate is 365.259641 . . Or more simplistically, the amount moved by Earth along its orbit from one full-moon to the next. Thus, from ratio (3u) the mean period of Earth in mean synodic months is:

$$
\begin{equation*}
\text { Period of revolution of Earth }=\frac{360^{\circ}}{u^{\prime}}=12 ; 22,7,51,53,40, \ldots \text { mean synodic months }=365.25952955 \text { days } \tag{3u}
\end{equation*}
$$

## IX. Multiple luni-solar extensions from Phi-series/synodic relation (1)

The simplicity of this relation permits similar derivations for the Draconic, Anomalistic and Nodal Cycles. The first pair include the mean synodic month (MSM) whereas the last cycle uses the Draconic (MDRA) and Tropical (MTROP) months:

$$
\begin{align*}
\text { Draconic Cycle } & =\frac{\text { MSM } \cdot \text { MDRA }}{\text { MSM }- \text { MDRA }}=346.6195761217 \ldots \text { days }  \tag{1dc}\\
\text { Anomalistic Cycle } & =\frac{\text { MSM } \cdot \text { MAN }}{\text { MSM }- \text { MAN }}=411.7805352634 \ldots \text { days }  \tag{1ac}\\
\text { Nodal Cycle } & =\frac{\text { MTROP } \cdot \text { MDRA }}{\text { MTROP }- \text { MDRA }}=18.6101191842 . . \text { years } \tag{1nc}
\end{align*}
$$

Here the nodal cycle is of potential interest in view of its association with lunar standstills in the first place and the apparent trouble the ancients took to delineate this phenomenon in the second, e.g. Chaco Canyon in the United States, Stonehenge in England and Callanish in Scotland. ${ }^{27}$

At this point Babylonian astronomy in general and the origins of these mean synodic arcs in particular begin to assume an unexpected level of importance despite almost universal dismissal of Babylonian methodology at the present time. For this reason it appears necessary to to offer an optional excursus after the Bibliography for Part V to explain the statements in Text-Fig 1concerning advanced knowledge of astronomy in the Old Babylonian period and other matters of concern.

## PART ONE: CLOSING REMARKS

Rejections: (1) Expansions of the Laws of planetary motion; (2) Benjamin Pierce's planetary framework Starting with Galileo and the velocity expansions of the laws of planetary motion described in the Excursus, the concern here is that while the present writer was merely a tertiary restorer, and as such did not expect much in the way of applause, it seemed a reasonable assumption that the extended version $T^{2}=R^{3}=V i,{ }^{6} R=V i{ }^{2}$ would at least take its place next to Kepler's twin parameter format $R^{3}=T$. ${ }^{2}$ And further, that variants of the former would simplify routine tasks, e.g., the calculation of angular momentum $L$, Table 1 mean velocities and the like. But this did not come to pass, and so it has remained ever since. On the other hand, modern science appears to have been able to function without such expansions, though not necessarily as well, it is suggested, had these velocity components also been incorporated.

But the real problem is not this historical item per se, but rather, that the same process and rapid dismissal was also applied to Benjamin Peirce's Fibonacci-based planetary framework with no replacement or improved version to take its place. And oddly, because of this situation which has remained unaddressed, humankind is now avidly searching for external planetary systems without any overall dynamic understanding of our own. Think not? Simply stated, no current model appears to exist which would, for example, provide the precise information and the theoretical basis for the possible existence of another planet interior to Mercury. Whereas, even in its initial form (sans intermediate intervals) this possibility was expressly incorporated in Pierce's initial approach, while in light of present concerns with Global Warming the possible intermediate location of Earth becomes more than a mere historical asterisk.

Weakened by special interests, discouraged by behavioural deficiencies and also impeded by disbelief, even the most fundamental question concerning whether Climate Change originates primarily from within, i.e., confines of planet Earth, or from without as an integral component of a larger System cannot be tackled adequately at present. Furthermore, what can be made of the location of Earth itself in the intermediate position between Venus and Mars, and what role might this apparent anomaly have played in global warming during the past, distant or otherwise?

All of which is further exacerbated by increasing population growth, unceasing deforestation, rapidly diminishing resources with warfare and mental illness also rising on a Global scale. Truly an Age of Disillusionment and concern. In the meantime the present inquiry turns next to the initial application of the Pierce Divisor approach to external planetary structures with or without the following suggested guidelines.

## PROVISIONAL GUIDELINES FOR EXTERNAL SYSTEMS

## Test Format, Phi-series Relations and Base Periods

Remaining with the order adopted by Peirce which commences with the outermost PLANET \#1 with the greatest period of revolution, moving inwards (by way of Synodic 2-1, then PLANET \#2, etc.), will generally involve three consecutive mean periods. All of which can be determined by the following Phi-series synodic relations if needed:

Relation (1) The Synodic mean between two bracketing periods of revolution, thus the product of the periods divided by their difference.
Relation (2). Relation (2) requires two adjacent periods above to generate the next value below, and thereafter generates all further lower periods if or as required.
Relation (4). The geometric mean of any pair of bracketing periods. Thus Relation (4 $\pm 1$ ), or simply Relation (4) as used.
Relation(4E) Relation ( $4 \mathrm{E} \pm 2$ ), Relation ( $4 \mathrm{E} \pm 3$ ), Relation ( $4 \mathrm{E} \pm 4$ ), Relation ( $4 \mathrm{E} \pm 5$ ) and Relation ( $4 \mathrm{E} \pm 6$ ). Such applications depend on specific prior restorations (in due order) above and below the target position(s).
Relation(4F) Relation (4F+3). Special case for PLANET \#2 only. Requires both the Base period and periods below \#2. This application serves to synchronize the restored periods at this point with the those of the divisor framework

The above relations are provided in Table 4 with the Fibonacci and the Lucas series in vertical and inverted form to match their inclusions in Tables 2a, 2 b and also the format adopted for exoplanetary structures.

Lastly, possible departures from the framework are included as variations which may be encountered among external systems. For similar systems, however
(a) Planets may occupy intermediate (synodic) locations, as in the case of Earth.
(b) Planets and adjacent synodic locations may be unfilled (i.e., absent), e.g., the Mars-Jupiter Gap.
(c) Departures from the theoretical framework, or (a) and (b) may indicate disrupted planetary systems.
(d) Planetary systems may possess residual Fibonacci indicators, as in the Solar System.
(e) Planetary systems may also possess residual Lucas indicators for the same reason as (d).

| Planets N Divisors Synodics \# (added) | PHI-SERIES RELATION (1), the Synodic mean: B = AC/\| ( $\mathrm{A}-\mathrm{C}$ )\|. <br> For any three successive Phi-series periods, A, B, C middle period (B) | Fibonacci series $1,1,2,3,5$ | Lucas series $1,3,4,7$ |
| :---: | :---: | :---: | :---: |
| 1 | is the product of the periods on either side divided by their difference. | 1 |  |
| Synodic 2-1 1 |  | 4181 | 778 |
|  | If two upper adjacent periods $\mathrm{A}, \mathrm{B}$ are known, the third and low | 22584 | 3571 |
|  | is the product of the two adjacent periods divided by their sum. | 31597 | 2207 |
|  |  | 5610 | 1364 |
| PLANET 36 | PHI-SERIES RELATION (4), the extended Geometric mean: $\mathrm{B}=\sqrt{ }(\mathrm{AC})$. | 8377 | 11843 |
| Synodic 4-3 9 | For any three successive periods, the middle period $(B)$ is the geometric mean of the periods on either side, as are the resulting periods for the | $13 \quad 233$ | 18521 |
| 15 | positions $\pm 2, \pm 3, . .4(4 \mathrm{E})$; relation $(4 \mathrm{~F}+3)$ pertains to Planet \#2 alone. | 144 | 29322 |
| Synodic 5-4 25 |  | 89 | 47199 |
| PLANET 540 |  |  | 76123 |
| Synodic 6-5 64 | Base period B1 is the period of the outermost planet as detected. Base Period B2 may be applicable if Synodic 2-1 is marginally > B1. |  | 76 |
| PLANET 6104 |  | 14 | 199 |
| Synodic 7-6 169 | BASE PERIOD B3. | $233 \quad 13$ | $322$ |
| PLANET 7273 | Approximate base periods (B3s) result from reversed procedures, i.e., the products of known periods and their assigned divisors. | $\begin{array}{ll} 233 & 13 \end{array}$ | $\begin{array}{ll}322 & 29 \\ 521 & 18\end{array}$ |
| Synodic 8-7 441 |  |  | 843 |
| PLANET 8714 |  |  | $\begin{array}{ll} 1364 & 7 \end{array}$ |
| Synodic 9-8 1156 | Approximate base periods (B4s) can be obtained from the averaged values of the available B3 products. | $159$ | 2207 |
| PLANET 91870 |  | 258 | 3571 |
| Synodic 10-9 3025 |  | 418 | 577 |
| PLANET 104895 | of the above prove to be applicable. | Fibonaci series | Lucas series |

Table 4. Divisor assignments, numerical series, Phi-series relations and conventions for base periods B1 thru B5.

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THE PIERCE FRAMEWORK AND EXTERNAL SYSTEMS

## EXTERNAL PLANETARY SYSTEMS

## Preliminary remarks and initial tests

This research has made use of the Exoplanet Orbit Database and the Exoplanet Data Explorer at exoplanets.org between 2018 and the present (2022). ${ }^{9}$ The research was placed on hold when the periods for the first and the last examples originally treated here - GJ 876 and HD 30177 - were changed drastically, which destroyed the associated analyses. These occurrences also caused a reevaluation of the available exo-planet data, resulting in a minor loss of confidence in the same, the coincidental changes notwithstanding. Nevertheless, both analyses have been omitted from the present discussion despite the further insights they provided.
What follows next is a reduced treatment presented on a take-it-or-leave it basis. I have assumed that periods of of "revolution","years" and "days" are just that as applied to the exoplanets and have proceeded accordingly. This is necessarily an initial attempt, but at least with standard procedures and a specific planetary framework as its basis.

## INITIAL TESTS

## 2:1 and 4:2 Octaves

Omitting single planets and deferring multiple configurations until later, dual configurations can include alternate planet-to-planet or synodic-to-synodic pairs, adjacent planet-synodic pairs, and lastly, widely separated pairs of either kind. In particular, the periods and the $2: 1$ ratios of outermost pairs (Planets 1 and 2 ) provide base period B1, plus from relation (1) a further base period B2 (Synodic 2-1). Since Synodic 2-1 is adjacent to Planet 2 theoretical periods below the latter can be derived from the successive applications of relation (2) if required. Either way, a full theoretical framework follows from the division of the selected base period by the standard divisors. A second set of data based on the known periods and Phi-series/synodic relations (1), (2) and (4) permits comparison with the latter and the determination of mean and individual errors. This can be useful if uncertainties arise, e.g., selection of outermost planets from either of the $2: 1$ or $4: 2$ ratios. The latter $-4: 1$ in terms of the theoretical framework and the fixed divisors - should not normally be present, but if detected in addition to the 2:1 ratio of the outermost pair, both can be incorporated in a possibly disturbed planetary framework, a primary example being HR 8799.

HR 8799 b-e (b, c and d detected in 2008, e in 2010$)^{12}$ Planets 1 and 2, Synodic 3-2 and Synodic 4-3, Base period B2 = 164,330 days (Synodic 2-1). Residual Fibonacci sequence 233-144-89-55-34-21-13-8-5-3-2-1-1 is completed at Planet 14. The restored/suggested overall planetary framework is discussed in detail later.

## Period ratios other than 2 : 1

Configurations below the three outermost planets, i.e., with period ratios other than 2:1 initially present difficulties, but as a beginning the ratios of detected periods can be compared with those of the Pierce framework divisors and where successful the results can be applied to generate $B 3$ base periods. Before describing the methodology one further element needs to be incorporated in the test procedures. As it turned out, the Fibonacci indicators introduced above became increasingly apparent during the inspection of planetary data, although without the Pierce planetary framework (especially relation (1) that generates the intermediate values) this association might well be dismissed as coincidence. In addition to HR 8799, examples of residual Fibonacci sequences are:

Kepler-460 c-b (2016) ${ }^{11}$ Planet \#2, adjacent Synodic 3-2 respectively, base period B3 $=881.5626$ days (2xKepler-460 c). with residual Fibonacci sequence: 55-34-21-13-8-5-3-2-1-1 completed at Planet 8.

Kepler-321 c-b (both planets detected in 2014). ${ }^{13}$ Kepler-321 c: 13.093921 days, Kepler-321 b: 4.915379 days. The two Kepler-321 planets detected to date have periods that round to alternate Fibonacci numbers 13 and 5. Phi-series/synodic relation (1) provides a synodic difference cycle between Kepler-321c and b of 7.869567 days which rounds in turn to 8 to complete the Fibonacci trio: 5-8-13. Although in years, this sequence is present in Solar System as 5 synodic periods of Venus in 8 years with 13 corresponding periods of revolution for this planet. Also, for Kepler-321, since relation (1) provides an adjacent period to that of Kepler-321b, successive applications of Phi-series/synodic relation (2) generate sequential periods below the latter which when rounded complete the Fibonacci sequence 13-8-5-3-2-1-1 at Planet 7. Up to this point the Pierce planetary framework and the associated divisors play no part. All that remains now are the positional assignments for the two planets and determination of the base period. This requires the ratio of the detected periods, assignment of the closest ratio from the divisor framework, and the selection of a B3 base period from the Divisor-Period products as follows:

The ratio of the Periods of Kepler-321 $c$ \& Kepler-321 $b=2.6638686$.
The ratio of the Divisors for Planet 4 \& Planet $5(40 / 15)=2.6666667$ (Fibonacci 8/3)

The divisors for Planet 4 and Planet 5 are 15 and 40, hence the following B3 base period options with mean errors from Planets 4 through 7 the deciding factor; differences are slight with the average period (B4) a third choice.

```
15x Kepler-321c = 196.408815 days, mean error: - 0.503%.
40x Kepler-321b = 196.615160 days, mean error: - 0.608%.
```

Thus the provisional assignments for Kepler $321 c$ and $b$ are Planets 4 and 5 with a B3 base period of $15 \times$ Kepler $321 c=$ 196.40882 days. Thereafter the Peirce planetary framework follows from the application of the standard divisors with relations (1) and (2) completing the residual Fibonacci sequence: 13-8-5-3-2-1-1 at planet 7.

| PLANETS N <br> Synodics | DIVISOR <br> (added) | PERIODS 1 <br> Base/Divisor | PERIODS 2 <br> B3:Restored |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| P3: Integers |  |  |  | | EXOPLANET |
| :---: |
| Kepler-321 | | \%Error |
| :---: |
| (Div:B3) |

Table 1. The Divisor framework, Kepler-321 Planets 4 and 5, Base period B3c.

HD 37605 b-c (both planets detected in 2014). ${ }^{4}$ HD 37605 c: 2,720 days. HD 37605 : 55.01307 days.
The suggestion of a Fibonacci presence provided by the near 13-day period of Kepler-321 c does not appear to be a coincidence, nor does it appear to be an isolated occurrence. For example, a similar presence is suggested by the lower period of two-planet HD37605, specifically, the approximate 55-day period of HD37650 b of 55.01307days. In this system the two detected periods are widely separated and both also occupy synodic rather than planetary positions. Nevertheless, though a variation from the successive planetary periods of Kepler-321, relations (1), (2) and (4) are equally applicable to both Planet and Synodic locations in the Phi-series planetary framework. Therefore sequential, multiple applications of these three relations permit the restoration of the seven periods between the two detected planets as shown in Table 7.

## ASSIGNMENTS

The ratio of the periods of HD $37605 c$ and HD $37605 b=49.442796$.
The ratio of the divisors for Synodic 4-3 and Synodic 8-7 = 49 (441/9).
The provisional base period (B3) is $9 x \mathrm{HD} 37605 c=24,480$ days.

| PLANETS N Synodics \# | DIVISORS (added) | PERIODS 1 <br> B3/Divisors | PERIODS 2 <br> B3r restoration | PERIODS 3 <br> B3r (rounded) | EXOPLANETS <br> HD 37605 | \%Error (B1: B3) |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| PLANET 1 | 1 | 24480 | 24480 | 24480 | $9 \times \mathrm{HD} 37605 \mathrm{c}$ | Base B3c |
| Synodic 2-1 | 1 | 24480 | 25405.81 | 25406 | Relation (1) 12th | 3.78\% |
| PLANET 2 | 2 | 12240 | 12467.16 | 12467 | $\operatorname{Rel}(4 \mathrm{~F}+3) 11$ th | 1.86\% |
| Synodic 3-2 | 4 | 6120 | 6349.265 | 6349 | $\operatorname{Rel}(4 \mathrm{E} \pm 3) 10 \mathrm{th}$ | 3.75\% |
| PLANET 3 | 6 | 4080 | 3899.066 | 3899 | $\operatorname{Rel}(4 \mathrm{E} \pm 4) 9$ th | -4.43\% |
| Synodic 4-3 | 9 | 2720 | 2720 | 2720 | HD 37605 c | 0.00\% |
| PLANET 4 | 15 | 1632 | 1646.780 | 1647 | Relation (1) 8th | 0.91\% |
| Synodic 5-4 | 25 | 979.2 | 1025.754 | 1026 | $\operatorname{Rel}(4 \mathrm{E} \pm 2) 2 \mathrm{nd}$ | 4.75\% |
| PLANET 5 | 40 | 612 | 621.0260 | 621 | Relation (1) 7th | 1.47\% |
| Synodic 6-5 | 64 | 382.5 | 386.8275 | 387 | $\operatorname{Rel}(4 \mathrm{E} \pm 4) 1 \mathrm{ST}$ | 1.13\% |
| PLANET 6 | 104 | 235.3846 | 234.1985 | 234 | Relation (1) 6th | -0.50\% |
| Synodic 7-6 | 169 | 144.8521 | 145.8786 | 146 | Rel ( $4 \mathrm{E} \pm 2$ ) 3rd | 0.71\% |
| PLANET 7 | 273 | 89.67033 | 88.31984 | 88 | Relation (1) 4th | -1.51\% |
| Synodic 8-7 | 441 | 55.51020 | 55.01307 | 55 | HD 37605 b | -0.90\% |
| PLANET 8 | 714 | 34.28571 | 33.89832 | 34 | Relation (2) 5th | -1.13\% |
| Synodic 9-8 | 1156 | 21.17647 | 20.97426 | 21 | " " | -0.95\% |
| PLANET 9 | 1870 | 13.09091 | 12.95715 | 13 | " " | -1.02\% |
| Synodic 10-9 | 3025 | 8.092562 | 8.009294 | 8 | " " | -1.03\% |
| PLANET 10 | 4895 | 5.001021 | 4.949701 | 5 | " " | -1.03\% |
| Synodic 11-10 | 7921 | 3.090519 | 3.059158 | 3 | " " | -1.01\% |
| PLANET 11 | 12816 | 1.910112 | 1.890646 | 2 | " " | -1.02\% |
| Synodic 12-11 | 20736 | 1.180556 | 1.168488 | 1 | " " | -1.02\% |
| PLANET 12 | 33552 | 0.729614 | 0.722164 | 1 | " " | -1.02\% |
|  |  |  |  |  | Mean value | 0.212\% |

Table 2. The Divisor planetary framework and HD 37605, Synodic 4-3 \& Synodic 8-7, Base B3c
The two widely separated planets HD $37605 c$ and $b$ with a period ratio of 49.442796 ( 2720 days/55.01307 days) are readily equated with the Peirce planetary framework and the ratio of the divisors for Synodic 4-3 and Synodic 8-7 (441 and 9 respectively). The lowest mean error is obtained from product of the period of HD 37605 c and the the divisor for Synodic 4-3 (9) = 24,480 days. The odd number of periods between the two detected planets permits the theoretical restoration of all periods between HD37605 $c$ and $b$ plus all those below the latter. This is feasible since relation (4) - the extended geometric mean - can be applied three times, first at the midpoint between the two planets (at Synodic 6-5) then twice more between two new mid-points to obtain the periods for Synodic 5-4 and Synodic 7-6. This fills three of the seven positions with those remaining determined by relation (1), including the period adjacent to HD 37605 b, thus permitting the generation of the periods for Planets 4,5 and 6 , and finally the completion of the residual Fibonacci sequence: 55-34-21-13-8-5-3-2-1-1 at Planet 12 by use of Relation (2). Another example involving both a Fibonacci indicator (5 versus 5.41608 days) and multiple applications of relation (4) and (2) is given below sans table.

HATS-59 b-c ${ }^{5}$ HATS-59 c: 1422 days, HATS-59 b: 5.41608 days.
Here the more widely separated periods of HATS-59 $c$ and $b$ (1422 and 5.41608 days respectively) can be assigned to Planet 1 and Planet 7 with a separation of eleven intervening periods. Again, the odd number permits multiple applications of relation (4) to determine the period of mid-point Planet 4 followed by mid-point periods on either side belonging to Synodic 3-2 and Synodic 6-5. In this instance there are two positions between the determined periods, not one, therefore relation (1) is not applicable. Instead, with the Pierce planetary framework available, the period of Planet 2 of 711 days can be introduced above Synodic 3-2 to allow relation (2) to end at Planet 9.

## ASSIGNMENTS

The ratio of the periods of HATS-59 $c$ and HATS-59 $b=262.551465$.
The ratio of the divisors for Planet 1 \& Planet $7(273 / 1)=273$ (3.83\%).
The provisional base period is B1, Planet 1, HATS-59 c (1422 days).
The residual Fibonacci presence 13-8-5-3-2-1-1 including HATS-59 $b$ is completed at Planet 9.

## Variations and Additions

Applying the above procedures to other systems brought to light additional numerical sequences including the double-Fibonacci series,. i.e., instead of Fibonacci 13-8-5, the sequence 26-16-10, etc. The second occurrence, the replacement of the residual Fibonacci series by the Lucas series (1-3-4-7-11-18-29-47-76-123,... etc.,) featured one common departure, namely the inclusion of the number 2 below the sequence 7-4-3. Examples of residual Lucas series present among the available external systems are:

Kapteyn's c-b (2014)6 Kapteyn's c: 124.54 days, Kapteyn's b: 48.616 days.
No series is initially indicated with the Lucas sequence only becoming apparent after the assignment of the base period and the derivation of the divisor framework. The two periods are again consecutive synodic locations, i.e., Synodic 5-4 and Synodic 6-5 respectively.

## ASSIGNMENTS

The ratio of the divisors for Synodic 5-4 and Synodic 6-5 $=2.56$. (square of 1.6 , or Fibonacci $5 / 3$ )
The ratio of the periods of Kapteyn's $c$ and Kapteyn's $b=2.5617081$.
The provisional base period is $\mathrm{B} 3=25 \times$ Kapteyn's $\mathrm{C}=3113.5$ days.
The residual Lucas series $18-11-7-4-3-\{2\}-1-1$ is complete at Planet 10 with $\{2\}$ anomalous.

Other external systems featuring Lucas sequences include the following, all with $\{2\}$ anomalous:
Nu Ophiuci $c-b(2010)^{7}$ Planets 1 and 3 respectively, Base period B1 = 3,186 days (as detected).
Residual Lucas sequence: 18-11-7-4-3- \{2\}-1-1 completed at Planet 10.
XO-2s $b-c(2014)^{8}$ Planets 3 and 5 respectively, Base Period $B 3=724.8$ days ( $6 \times$ XO-2s c).
Residual Lucas sequence: 18-11-7-4-3-\{2\}-1-1 completed at Synodic 9-8.
Kepler-49 c-b (2012) ${ }^{9}$ Synodic 3-2 and Planet 3 respectively, Base period B3 $=43.6517372$ days ( $4 x$ Kepler-49 c). Residual Lucas sequence: 11-7-4-3- \{2\}-1-1 completed at Synodic 8-7
Kepler-198 $c-b(2014)^{10}$ Planets 3 and 4 respectively, Base period B3 $=273.404496$ days ( $6 x$ Kepler-198 c). Residual Lucas sequence: 29-18-11-7-4-3- \{2\}-1-1 completed at Synodic 8-7.
Kepler-396 $c-b(2014)^{11}$ Planet 2 and Synodic 3-2 respectively, Base period B3 $=177.01$ days ( $2 x$ Kepler-396 c). Residual Lucas sequence: 29-\{17\}-11-7-4-3- \{2\}-1-1 completed at Planet 7 with \{17\} for Lucas number 18.
HD $60532 b-c(2008)^{12}$ Planets 2 and 3 respectively, Base period B3 $=1214.12$ days ( $2 x \mathrm{HD} 60532 \mathrm{c}$ ).
Residual Lucas sequence: 76-47-29-18-11-7-4-3-\{2\}-1-1 completed at Planet 9.
HD 163607 b-c (2011) ${ }^{13}$ Planets 3 and 6 respectively, Base Period B3 = 7,884 days ( $6 x$ HD 163607 c).
Residual Lucas sequence: 76-47-29-18-11-7-4-3-\{2\}-1-1 completed at Planet 11.
Finally, partial residual sequences, i.e., confined to three consecutive values occur in some instances, while other systems, e.g., TRAPPIST-1 treated next have no immediately discernable sequence.

```
TRAPPIST-1 (b-g detected in 2016,'14 TRAPPIST-1 }h\mathrm{ detected in 2017). . }\mp@subsup{}{}{15
    TRAPPIST-1h: 18.767 days }\mp@subsup{}{}{26
    TRAPPIST-1g: 12.35294 days
    TRAPPIST-1f: 9.206690 days
    TRAPPIST-1e: }6.099615\mathrm{ days
    TRAPPIST-1d: 4.049610 days
    TRAPPIST-1c: 2.4218233 days
    TRAPPIST-1b: 1.51087081 days
Base period B3, Planet #1: 36.82876 days (4x TRAPPIST-1f)
```

Initially the period of TRAPPIST-1h was thought to range between14 and 35 days with the absence of a precise base period preventing generation of a divisor-based planetary framework. Nevertheless the latter still appears to be present in the structure of TRAPPIST-1 as indicated by the sequential twinned reduction ratios for both two-third ratios and also the upper three-fifth ratio. Thus the Pierce ratios have a key role to play in the present example. In more detail, commencing with the 9.206690 -day period of TRAPPIST- $1 f$ the successive reduction ratios generate ordered approximations for the periods of the remaining four TRAPPIST-1 planets1e through1 $b$. Furthermore, the application of Phi-series relation(4) to the periods of TRAPPIST-1f and TRAPPIST-1d results in a period of 6.081193 days versus the 6.0992672-period of TRAPPIST-I e.

Although the twinned Pierce ratios appear to be reflected in a substantial part of the structure of TRAPPIST-1 it appears that with respect to the theoretical framework the five adjacent planets TRAPPIST-1f through 1b occupy consecutive planetary/synodic positions. Thus while the Solar System has one planet (Earth) in a synodic location between adjacent planets, TRAPPIST-1 appears to have at least three in ordered succession, two of which can be approximated (if not confirmed) by relation(1). In short, the general synodic formula can be applied to the mean periods of revolution of TRAPPIST-1c and $1 e$ to approximate the difference period in the current location of 1d, and and similarly applied to TRAPPIST $1 b$ and $1 d$ for the difference period in the current location of $1 c$.

| PLANETS N Synodics \# | RATIOS <br> (Pierce) | DIVISOR (added) | DIVISOR RATIO <br> Results/Ratios | B1-RATIOS <br> B1s/Divisors | PERIODS 1 <br> B3/Divisors | PERIODS 2 <br> B1s(actual) | EXOPLANETS <br> TRAPPIST-1 | $\begin{gathered} \text { \%ERR } \\ (\mathrm{B} 1: \mathrm{B} 3) \end{gathered}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| PLANET 1 | 1/1 | 1 | (1/1) |  | 36.826760 |  | 4x TRAPPIST-1f | Base 3 |
| Synodic | 1/1 | 2 | 2 (2/1) | 1.9623147 | 18.413380 | 18.767 | TRAPPIST-1h | 1.92\% |
| PLANET ? | 1/2 | 3 | 3 (3/1) | 2.8912142 | 12.275587 | 12.35294 | TRAPPIST-19 | 0.63\% |
| Synodic | 1/2 | 4 | 4 (4/1) | 4 | 9.20669 | 9.20669 | TRAPPIST-1f | 0.00\% |
| PLANET 3 | 2/3 | 6 | 3 (6/2) | 6.0378991 | 6.1377933 | 6.0992672 | TRAPPIST-1e | -0.63\% |
| Synodic 4-3 | 2/3 | 9 | 2.25 (9/4) | 9.1683135 | 4.0918622 | 4.0167431 | TRAPPIST-1d | -1.84\% |
| PLANET 4 | 3/5 | 15 | 2.5 (15/6) | 15.206213 | 2.4551173 | 2.4218233 | TRAPPIST-1c | -1.36\% |
| Synodic 5-4 | 3/5 | 25 | 2.777* (25/9) | 24.374526 | 1.4730704 | 1.5108708 | TRAPPIST-1b | 2.57\% |
| PLANET 5 | 5/8 | 40 | 2.666*(40/15) | 39.580739 | 0.9206690 | 0.9304212 | (Rel.2) | 1.06\% |
| Synodic 6-5 | 5/8 | 64 | 2.56 (64/25) | 63.955265 | 0.5754181 | 0.5758206 |  | 0.07\% |
| PLANET 6 | 8/13 | 104 | 2.6 (104/40) | 103.53600 | 0.3541035 | 0.3556904 |  | 0.45\% |
| Synodic 7-6 | 8/13 | 169 | 2.6406 (169/64) | 167.49127 | 0.2179098 | 0.2198727 | " " | 0.90\% |
| PLANET 7 | 13/21 | 273 | 2.625 (273/104 | 271.02727 | 0.1348966 | 0.1358784 |  | 0.73\% |
| Synodic 8-7 | 13/21 | 441 | 2.609 (441/169) | 438.51854 | 0.0835074 | 0.0839799 |  | 0.57\% |
| PLANET 8 | 21/34 | 714 | 2.615 (714/273) | 709.54581 | 0.0515781 | 0.0519019 |  | 0.63\% |
|  |  |  |  |  |  |  | Mean error: | 0.407\% |

Table 3. The Pierce planetary framework, TRAPPIST-1, compressed adjacent planets, Base period B3
Irrespective of additional complications that arise from the dual occurrence of TRAPPIST-1d and $1 c$ in both synodic computations the first two-thirds reduction ratio nonetheless serves to synchronize TRAPPIST- $1 e$ with divisor planet \#3 and therefore all the remaining positions. At which point the simplest option for a theoretical base period is to reverse standard procedures and use the products of the known periods and their associated divisors to generate B3 estimates, with TRAPPIST- $1 f(4 x=36.82676$ days) the closest to the mean value.

More recently, ${ }^{16}$ a period of 18.767 days has been deduced for TRAPPIST- $1 h$ with a resulting Synodic 2-1 interval of 36.14366 days between the latter and the 12.35294 -day period of TRAPPIST- 1 g . The last period could be applied as a provisional base period B2. Lastly, though the12.35294-day period of TRAPPIST-1 $g$ appears to be anomalous, the synodic period between this value and any of the 36-day estimates for base period B3 results in values between 18.356 and 18.767 days. All of which raises the possibility that if disruptions of TRAPPIST- 1 may have occurred, that they might have involved the two outermost planets. If so there would be no perceptible gap per se, but absence of the outermost planet (or both) and possible readjustments by the others. Whether this scenario would be drastic enough to cause TRAPPIST-1 $b$ thru $1 f$ to occupy adjacent sidereal/synodic locations en masse or cause the anomalous period of $1 g$ is another matter. Then again, this scenario may also be a more natural occurrence with compression a component of later phases in the life-cycle of this particular System itself.

## Further candidates for compressed systems

Additional systems with apparent planet/synodic compression and residual Fibonacci/Lucas sequences include:
YZ Cet $d$ - $c-b$ (2017) ${ }^{17}$ Synodic 3-2, Planet 3, Synodic 4-3. Base B3 $=18.62508$ days
( 4 x YZ Cet $d$ ). Residual Fibonacci sequence: 5-3-2-1-1 completed at Synodic 5-4.
Kepler-23 $d-c-b(b-c$ 2012, $d 2014){ }^{18}$,Synodic 3-2, Planet 3, Synodic 4-3, Planet 4. Base B3 $=61.097216$ days
( $4 x$ Kepler-23 d). Residual Lucas sequence: 11-7-4-3- \{2\}-1-1 completed at Planet 6.
Kepler-37 d-c-b (2016) ${ }^{19}$ Synodic 4-3, Planet 4, Synodic 5-4. Base B3 $=358.129683$ days.
( $9 x$ Kepler-37 d). Residual Fibonacci sequence: 21-13-8-5-3-2-1-1 completed at Synodic 8-7.
Kepler-107 e-d-c-b (2014) ${ }^{20}$ Planet 2, Synodic 3-2, Planet 3, Synodic 4-3. Base B3 $=29.498352$ days
( $2 x$ Kepler-107 e). Residual Fibonacci sequence: 8-5-3-2-1-1 completed at Planet 5.
Kepler-184 d-c-b (2014) ${ }^{21}$ Synodic 5-4, Planet 5, Synodic 6-5. Base B3 $=203.304324$ days
( $25 x$ Kepler-184 d). Residual Lucas sequence: 11-7-4-3- \{2\}-1-1 completed at Synodic 9-8.

Kepler-208 e-d-c-b (2014) ${ }^{22}$ Synodic 3-2, Planet 3, Synodic 4-3, Planet 4. Base B3 $=65.0783$ days ( $4 x$ Kepler-208d). Residual Lucas sequence: 11-7-4-3- \{2\}-1-1 completed at Planet 6.
Kepler-295 d-c-b (2014) ${ }^{23}$ Planet 3, Synodic 4-3, Planet 4. Base B3 $=203.304324$ days ( $6 x$ Kepler-295 d). Residual Fibonacci sequence: 34-\{22\}-13-8-5-3-2-1-1 completed at Planet 7.
Kepler-374 $d-c-b(2014)^{24}$ Planet 3, Synodic 4-3, Planet 4. Base B3 $=30.169314$ days ( $6 x$ Kepler-374 d). Residual Fibonacci sequence: 5-3-2-1-1 completed at Planet 5.
Kepler-446 d-c-b (2014) ${ }^{25}$ Synodic 4-3, Planet 4, Synodic 5-4. Base B3 $=46.340289$ days ( $9 x$ Kepler-446 d). Residual Fibonacci sequence: 5-3-2-1-1 completed at Synodic 6-5.
Kepler-758 e-d-c-b (2016) ${ }^{26}$, Planet 4, Synodic 5-4, Planet 5, Synodic 6-5. Base B3 $=307.4493$ days (15xKepler-758d). Residual Fibonacci sequence: 8-5-3-2-1-1 completed at Synodic 8-7.

HR 8799 (HR 8799 b, c and d detected in 2008, HR 8799 e detected in 2010. ${ }^{2}$
Initially detected as a three-planet system, HR 8799 has known ${ }^{3} 1: 2$ and 1:4 resonances already subject to detailed analyses. ${ }^{27,28}$ The theoretical planetary framework is augmented by a possible 1:9 resonance following the discovery in 2010 of a fourth planet (HR 8799 e) with a period of 18,000 days ${ }^{38}$ versus 18,250 days for the associated resonance.

ASSIGNMENTS (MEAN PERIODS: Days/Julian Years)
HR $8799 b: 164,250$ days/449.691991786 years. HR $8799 \mathrm{c}: 82,145$ days/224.900752909 years.
HR $8799 \mathrm{~d}: 41,054$ days $/ 112.399726215$ years. HR 8799 e $: 18,000$ days/49.2813141684 years.
The ratio of the Divisors for Planet 1 and Planet $2(2 / 1)=2.00000$; second octave (HR 8799 c and $d$ ) is also present. Ratio of the periods of HR $8799 b$ and $c$ is 1.999613 ; the $4 / 2$ ratio of the second octave HR $8799 c$ and $d=2.000901$. The provisional base period (B2) is Synodic 2-1(164330.019 days) between HR $8799 b$ and $c$ (lowest mean error).

| PLANETS N Synodics \# | RATIOS <br> (Pierce) | DIVISORS <br> (added) | PERIODS 1 <br> B2/Divisors | PERIODS 2 <br> B1 Actual | PERIODS 3 <br> B1/(Rounded | EXOPLANETS <br> HR 8799 | \%ERROR <br> (B1: B2) |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| PLANET 1 | 1/1 | 1 | 164330.019 | 164250 | 164250 | HR 8799 b | -0.049\% |
| Synodic 2-1 | 1/1 | 1 | 164330.019 | 164330.019 | 164330 | (Rel. 1) | 0.000\% |
| PLANET 2 | 1/2 | 2 | 82165.0097 | 82145 | 82145 | HR 8799 c | -0.024\% |
| Synodic 3-2 | 1/2 | 4 | 41082.5049 | 41054 | 41054 | HR 8799 d | -0.069\% |
| PLANET 4 | 2/3 | 6 | 27388.3366 | 27373.443 | 27373 | (Rel. 2) | -0.054\% |
| Synodic 4-3 | 2/3 | 9 | 18258.8911 | 18000 | 18000 | HR 8799 e | -1.418\% |
| PLANET 4 | 3/5 | 15 | 10955.3346 | 10859.2591 | 10859 | (Rel. 2) | -0.877\% |
| Synodic 5-4 | 3/5 | 25 | 6573.20078 | 6733.10057 | 6773 |  | 3.041\% |
| PLANET 5 | 5/8 | 40 | 4108.25049 | 4171.35627 | 4171 | " " | 1.536\% |
| Synodic 6-5 | 5/8 | 64 | 2567.65655 | 2581.49088 | 2581 | " " | 0.539\% |
| PLANET 6 | 8/13 | 104 | 1580.09634 | 1594.63378 | 1595 | " " | 0.920\% |
| Earth/Syn 7-6 | 8/13 | 169 | 972.366979 | 985.730287 | 986 | " " | 1.374\% |
| PLANET 7 | 13/21 | 273 | 601.941463 | 609.169393 | 609 | " | 1.201\% |
| Synodic 8-7 | 13/21 | 441 | 372.630430 | 376.498114 | 376 | " " | 1.038\% |
| PLANET 8 | 21/34 | 714 | 230.154089 | 232.686099 | 233 |  | 1.100\% |
| Synodic 9-8 | 21/34 | 1,156 | 142.153996 | 143.808516 | 144 | " " | 1.164\% |
| PLANET 9 | 34/55 | 1,870 | 87.8770158 | 88.8784095 | 88 |  | 1.140\% |
| Synodic 10-9 | 34/55 | 3,025 | 54.3239734 | 54.9299112 | 55 |  | 1.115\% |
| PLANET 10 | 55/89 | 4,895 | 33.5709948 | 33.9485443 | 34 |  | 1.125\% |
| Synodic 11-10 | 55/89 | 7,921 | 20.7461204 | 20.9813561 | 21 |  | 1.134\% |
| PLANET 11 | 89/144 | 12,816 | 12.8222550 | 12.9671908 | 13 |  | 1.130\% |
| Synodic 12-11 | 89/144 | 20,736 | 7.92486591 | 8.01416473 | 8 |  | 1.127\% |
| PLANET 12 | 144/233 | 33,552 | 4.89777121 | 4.95302617 | 5 | " " | 1.128\% |
| Synodic 13-12 | 144/233 | 54,289 | 3.02694873 | 3.06113853 | 3 |  | 1.130\% |
| PLANET 13 | 233/377 | 87,841 | 1.87076672 | 1.89188765 | 2 |  | 1.129\% |
| Synodic 14-13 | 233/377 | 142,129 | 1.15620330 | 1.16925087 | 1 |  | 1.128\% |
| PLANET 14 | 377/610 | 229,970 | 0.71457155 | 0.72263678 | 1 |  | 1.129\% |
|  |  |  |  |  |  | Mean error: | 0.809\% |

Table 4. The Pierce planetary framework and HR 8799; Planets 1 and 2, Synodics 3-2 and 4-3. Bases B1 and B2.

## HR8799 AS A DISTURBED PLANETARY SYSTEM

As in the case of the Solar System, base period B2 (164330.019 days, Synodic 2-1 between HR 8799 c and HR 8799 b) is marginally greater than base period B1 ( 164,250 days). This, allied with 1:4 and 1:9 resonances recognizable as squares belonging to two successive synodic positions in the divisor framework suggests that HR8799 may also be a disrupted system as seen in Table 4 and Figure 1 below. If so, it is perhaps possible that HR $8799 d$ and HR $8799 e$ may currently be occupying Synodic 4-3 and Synodic 3-2 locations resulting from outward orbital shifts of divisor planets \#3 and \#4. In which case theoretical mean periods of revolution for the latter pair can be approximated by either successive applications of relation (2) to HR 8799 c and HR 8799 d, and (or) the application of divisors 1 to 40 to base period B2. Whether a theoretical planet at or near position \#5 ( $\sim 4.5$ HR 8799 standard mass?) suffered a catastrophic demise is hypothetical, but still a possibility which can be considered further in terms of the debris field in the inner region of HR 8799 discussed by Moore and Quillen (2013), ${ }^{29}$ Contro et al, (2014) ${ }^{30}$ and Contro et al, (2016) ${ }^{31}$. In particular, the theoretical distance for possibly defunct HR_8799_5 at $\sim 5.02$ a.u. is situated reasonably close to the "inner and outer edges, located at $\sim 6$ and $\sim 8$ au," of the debris belt discussed by the latter authors.


Fig 1. HR $8799 b$ to $e$ and departures from the Pierce planetary framework for planets 1 through 5.

In any event, if this scenario is valid there is a distinct similarity between HR 8799 and the Solar System, sincefor whatever reasons-the fifth planet in both systems (counting inwards) can be considered to be absent. Also, in addition to the known planetary resonances, similarities between the two systems, especially with respect to the gas giants Jupiter and Saturn have already been noted by Fabrycky and Murray-Clay (2010). ${ }^{31}$ In fact, the similarity between the two may be greater than already suspected, as shown in log-linear Figure 2 with the Solar System in an eight-planet configuration and HR 8799_1_9 as a theoretical nine-planet system. Or, as a substitute, depending on what might have taken place and the original mass of planet HR 8799_5, major compensatory adjustments that may have occurred in another eight-planet system.

The linkage between the divisors from the Fibonacci-based Peirce approach, the Phi-series planetary framework and the Lucas series - all with respect to unity and the mean parameters of Earth - is also shown in Figure 2. Here the eight-planet Solar System from Saturn to Mercury is compared to a theoretical inward extension of HR 8799 (HR 8799_4 thru HR 8799_9) with the latter represented as a nine-planet system.


Fig. 2. HR 8799_4 to HR8799_9, extended Pierce resonances and the Solar System from Saturn to Mercury.
In the above configuration Solar System base B2 is 171.4442890 years, HR 8799 base B2 is 449.9110736 years and HR 8799 base B1 is 449.69199818 years. The similarity between the two Systems, the Lucas Series and lower Phi-series gives rise to the following relations involving the limiting Pierce reduction ratio ${ }^{-2}(0.38196601125)$ and reciprocal ${ }^{2}$ (2.61803398875), the outward Pheidian constant for the periods of revolution, thus:

Solar System, B2 $=171.4442890$ years
Base B1,HR 8799. ${ }^{-2}=171.8507380$ years
Base B2, HR 8799. ${ }^{-2}=171.7670564$ years

$$
\begin{aligned}
{ }^{2} \cdot \text { B2, Solar System } & =448.8469772 \text { years } \\
\text { Base B1, HR 8799 b } & =449.6919918 \text { years } \\
\text { Base B2, HR } 8799 & =449.9110736 \text { years }
\end{aligned}
$$

and the following comparison:

| PLANETS N | DIVISORS | Sol. System II | HR 8799 | LUCAS | Phi-Series ${ }^{\times}$ | Exp. |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Synodics \# | 1-714 | B2/Div. 1-714 | B2/ ${ }^{2}$ | N | (Periods/years) | X |
| NEPTUNE 1 | 1 | 171.444290 | 171.767283 |  | 199.0050249 | 11 |
| Synodic 2-1 | 1 | 171.444290 | 171.767283 |  | 122.9918694 | 10 |
| URANUS 2 | 2 | 85.7221448 | 85.8836413 |  | 76.01315562 | 9 |
| Synodic 3-2 | 4 | 42.8610724 | 42.9418207 |  | 46.97871376 | 8 |
| SATURN 3 | 6 | 28.5740483 | 28.6278804 |  | 29.03444185 | 7 |
| Synodic 4-3 | 9 | 19.0493655 | 19.0852536 |  | 17.94427191 | 6 |
| JUPITER 4 | 15 | 11.4696193 | 11.4511522 | 11 | 11.09016994 | 5 |
| Synodic 5-4 | 25 | 6.85777158 | 6.87069131 | 7 | 6.854101966 | 4 |
| (MJ-Gap) 5 | 40 | 4.28610724 | 4.29418207 | 4 | 4.236067977 | 3 |
| Synodic 6-5 | 64 | 2.67881702 | 2.68386379 | 3 | 2.618033988 | 2 |
| MARS 6 | 104 | 1.64850278 | 1.65160849 | (2) | 1.618033988 | 1 |
| Earth/Syn 7-6 | 169 | 1.01446325 | 1.01637445 | 1 | 1.000000000 | 0 |
| VENUS 7 | 273 | 0.62800106 | 0.62918419 | 1 | 0.618033988 | -1 |
| Synodic 8-7 | 441 | 0.38876256 | 0.38949497 | - | 0.381966011 | -2 |
| MERCURY 8 | 714 | 0.24011805 | 0.24057042 | - | 0.236067977 | -3 |

Table 5. Solar System and HR 8799 with emphasis on the Lucas \& Phi-Series.
All of which is encouraging enough to precipitate a return to the Solar System and an investigation of these two fundamental constants and related variants with respect to real-time varying motions of the planets, commencing with those of Jupiter and Saturn followed by the terrestrial planets and the remaining two superior planets.

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REAL-TIME MOTION IN THE SOLAR SYSTEM AND THE GOLDEN SECTION

## REAL-TIME MOTION IN THE SOLAR SYSTEM AND THE GOLDEN SECTION <br> The sequential pheidian constants ${ }^{-3},{ }^{-2},{ }^{-1}, 0,1^{1},{ }^{2}$ and the Solar System

Jupiter-Saturn mean synodic ratios and the primary constants, ${ }^{2}$.
As seen in Table 3s both the Lucas and the Fibonacci series appear to be present, albeit with respect to the mean periods and motions, not the more complex varying velocities associated with the elliptical orbits of the planets. To include the latter what follows next initially concentrates on Jupiter and Saturn and the mean synodic data for these two planets calculated with respect to the motion of Earth (unity) as given in most modern tables. Thus for Jupiter a mean synodic period $(S)$ of 1.0921105 years, during which the planet moves $33.15978173^{\circ}$, i.e., the mean synodic arc ( $u$ ). During the same interval Earth concurrently completes $360^{\circ}+u=393.1597817291^{\circ}$, i.e., one year plus ( $u$ ). The same procedure for Saturn yields in turn a mean synodic period of 1.03518213 years, $372.665567397^{\circ}$ of sidereal motion for Earth and a corresponding mean synodic arc ( $u$ ) of $12.665567497^{\circ}$.

However, the ratio of these two accurate mean synodic arcs turns out to be of major significance, i.e.,

$$
33.159781729^{\circ} / 12.665567497^{\circ}=2.61810472 \text { vs }^{2}=2.61803398875 \text { or Phi-series relation (8) }
$$

whereas the ratio of the mean synodic arcs for these two planets applied in Babylonian astronomical cuneiform texts of the of the Seleucid Era ${ }^{2}\left(-310\right.$ to 75 ) of $33 ; 8,45^{\circ}{ }^{3}$ and $12 ; 39,22,30^{\circ 4}$ produces a slightly less accurate, but nonetheless similar result, i.e., (as shown earlier):

$$
33.1458333^{*} 0 / 12.65625^{\circ}=2.61893004 \text { vs }^{2}=2.61803398875 \text { Phi-series relation (8) }
$$

Further investigation of the heliocentric methodology inherent in Babylonian astronomy next involves a parameter $P$ and the following triple relation based on the mean sidereal period $T$ and the mean synodic arc $u:{ }^{5}$

$$
\begin{equation*}
P=\left(360^{\circ} / u\right)=T-1, T=P+1 \tag{13}
\end{equation*}
$$

resulting in ratios between modern $T$ - 1 periods for Saturn and Jupiter (Saturn $P$ and Jupiter $P$ respectively) of:
Saturn $P / J$ upiter $P=2.61810472$ vs 2.618033989 (relation 7, the planet-to-planet increment)
Jupiter $P /$ Saturn $P=0.38195569$ vs 0.381966011 (limiting constant, Pierce planet-to-planet ratios)

## Jupiter-Saturn real-time test formulas

Next, although secondary in nature, the $P$-periods are also revolutions with the general synodic formula applicable here also. Thus, with respect to the Jupiter-Saturn synodic difference cycle (hereafter SD1), the $P$-variant (SD1P) provides two more recognizable results:

$$
\begin{align*}
& \text { SaturnP/SD1P }=1.61810472 \text { vs } 1.618033989 \text { (relation } 8 \text {, the planet-synodic increment) }  \tag{16}\\
& S D 1 P / \text { Jupiter } P=1.61800697 \text { vs } 1.618033989 \text { (relation } 8 \text {, the planet-synodic increment) } \tag{17}
\end{align*}
$$

with the primary Phi-series constants and ${ }^{2}$ also testable against the real-time Solar System in the form:

$$
\begin{align*}
& \text { Ratios of adjacent periods, both } T_{N} \text { and } S_{N}=1.61803398875 \quad \text { (Phi-series relation 8) }  \tag{18}\\
& \text { Ratios of alternate periods, both } T_{N} \text { and } S_{N}=2.61803398875 \quad \text { (Phi-series relation 7) } \tag{19}
\end{align*}
$$

## Synodic difference cycles SD1: Jupiter - Saturn, SD2: Saturn - Uranus, and SD3: Uranus - Neptune

Finally, the increasing departures from the Solar System beyond Saturn notwithstanding, the initial tests for realtime motion can be investigated with respect to the synodic difference cycles between the four adjacent superior planets. Here initial emphasis is placed on relations (20) and (21); the Phi-series departures from the Peirce divisors inherent in relation (22) are discussed later:

$$
\begin{array}{ll}
\text { Jupiter-Saturn synodic SD1 } f(\mathrm{t}): & \frac{\text { Saturn } f(\mathrm{t}) \cdot \operatorname{Jupiter} f(\mathrm{t})}{\text { Saturn } f(\mathrm{t})-\operatorname{Jupiter} f(\mathrm{t})}=\operatorname{SC4-3} \\
\text { Saturn-Uranus synodic SD2 } f(\mathrm{t}): & \frac{\operatorname{Uranus} f(\mathrm{t}) \cdot \operatorname{Saturn} f(\mathrm{t})}{\operatorname{Uranus} f(\mathrm{t})-\operatorname{Saturn} f(\mathrm{t})}=S C 3-2 \\
\text { Uranus-Neptune synodic SD3 } f(\mathrm{t}): & \frac{\text { Neptune } f(\mathrm{t}) \cdot \operatorname{Uranus} f(\mathrm{t})}{\text { Neptune } f(\mathrm{t})-\operatorname{Uranus} f(\mathrm{t})}=\text { SC2-1 } \tag{22}
\end{array}
$$

## Real-time planetary motion and period ratios in the Solar System

## Methodology

In 1986, following the general availability of personal computers in the early1980s, Pierre Bretagnon and Jean-Louis Simon published Planetary Programs and Tables from -4000 to+2800: Tables for the Motion of Uranus and Neptune from +1600 to 2800 . $^{1}$ The programs generate geocentric planetary positions for single dates by initially calculating the heliocentric radius vector $R$ in a.u., the heliocentric longitude $L$ in radians and heliocentric latitude $B$. Although geocentric conversions and corrections follow, the heliocentric data proves to be entirely sufficient for the present purpose since the corresponding periods and velocities can be obtained from the instantaneous heliocentric radius vectors via the harmonic law and velocity components of the same. Following a number of minor adaptations to generate heliocentric time-series outputs, the varying radius vectors and the corresponding sidereal and synodic periods, velocities and ratios for the superior planets were generated in sets of 36,525 data points per century, i.e., intervals of one Julian day to synchronize with the methodology instituted by Bretagnon and Simon. Much shorter intervals with smaller test periods (e.g., 6-hourly data) were adopted for Mars (planet \#6), for Earth (synodic cycle SC 7-E), Venus (\#7), Mercury (\#8) and synodic cycle SC 8-7 between the latter pair.

## Initial testing

Relations 20 through 22 involve the major Jupiter-Saturn synodic difference cycle (SD1), the secondary SaturnUranus synodic cycle (SD2) and the third major synodic difference cycle between Uranus and Neptune (SD3). In terms of real-time motions these relations and variants can be tested against the Phi-series as follows:

Relative motions of Jupiter, Synodic cycle SD1, Saturn and the constant $=\mathbf{1 . 6 1 8 0 3 3 9 8 8 7 5 .}$
Commencing on Julian day JD2451544.5 (January 1st 2000) and ending on Julian day JD488069.5 (January 1st 2100) both the $P$ and $T$ ratios of interest were plotted against Phi-series relation (8), i.e., the constant (1.61803398875). Thus data for the ratios (SaturnP/SD1P), (SD1P/JupiterP) and (SaturnP/JupiterP) were generated along with a second control set from the unmodified ratios, (SaturnT/SD1), (SD1/JupiterT) and (SaturnT/JupiterT). Initial tests show that despite consistent differences in the amplitudes and times the intercepts are nevertheless pheidian. In other words, both sets intersect at the target constant (1.61803398875) with finer definition available if required. The initial results also simplified matters considerably since further testing could proceed using real-time parameters as shown in Figure 1a (both sets) and thereafter standard $T$-values for the configurations shown in Figures 1 b through 6 c .


Fig. 1a. Daily Tand P ratios for Saturn, SD1 and Jupiter, 2000 CE to 2100 CE, $k==1.61803398875$.
Figure 1b also includes the corresponding waveform for the sixth root of the generating Phi-series exponent (6) of the Jupiter-Saturn synodic difference cycle SD1. The similar waveforms for Jupiter and Saturn with generating exponents of 5 and 7 respectively have been omitted for clarity.


Fig. 1b. Daily Tratios for Saturn, SD1and Jupiter, plus SD $1^{1 / 6}$ from 2000 CE to 2100 CE, $k==1.61803398875$.

The relative motions of Jupiter (T), Synodic cycle SD1, Saturn (T), and constant ${ }^{-2}$ ( 0.38196601125 ). Next, the two SD1-associated $T$-ratios were reduced by the exponent -2 for comparison with the limiting value of Peirce's planet-to-planet reduction ratios ( $k=^{-2}=0.38196601125$ ). The Saturn/Jupiter ratios were reversed sans exponentiation since the Jupiter $T$ / Saturn $T$ ratio is also the mean synodic velocity of Jupiter with respect to Saturn for the Phi-series planetary framework (SD1Vr). Relative orbital velocities (Vr) were included for both planets, and ${ }^{-2}$ ( 0.38196601125 ) examined in terms of the parameters associated with the Inferior planets Mercury and Venus.


Fig. 2. Daily period ratios \& orbital velocities (Vr): Jupiter T, SD1 \& Saturn T, 2000-2100 CE, $k={ }^{-2}=0.38196601125$.

## The Terrestrial planets and synodic location of Earth between Mars and Venus

Although Earth appears to be occupying the synodic location SC7-6 between Mars and Venus, the synodic period SC7-E between the latter and Earth is atypically greater than that of Earth itself. In other words, the intermediate, synodic period has a mean value that is greater than that of the outermost planet, i.e., 1.5986495 years, thus close to Fibonacci $8 / 5=1.6$. Also of interest is the occurrence of ${ }^{-2}$ for the mean heliocentric distance ( $R$ ) of Mercury, the mean synodic period $(S)$ of Mercury with respect to Venus (SC8-7) and also relative velocity (Vr) of Jupiter-Saturn SD1.

| PLANETS N Synodics \# | $\begin{aligned} & \hline x=\text { Phi-series }(T) \\ & x \quad \text { (Years) } \end{aligned}$ |  | Phi-series ( R ) <br> Distance (a.u.) | Phi-series (Vi) Inverse Velocity | $\begin{gathered} \hline \hline \text { Phi-series (Vr) } \\ \text { Velocity (ref.1) } \end{gathered}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| Mars 6 | 1 | 1.618033989 | 1.378240772 | 1.173984997 | 0.851799642 |
| Earth/Syn 7-E | 0 | 1.000000000 | 1.000000000 | 1.000000000 | 1.000000000 |
| Venus 7 | -1 | 0.618033989 | 0.725562630 | 0.851799642 | 1.173984997 |
| Syn SC8-7 | -2 | 0.381966011 | 0.526441130 | 0.725562630 | 1.378240772 |
| Mercury 8 | -3 | 0.236067978 | 0.381966011 | 0.618033989 | 1.618033989 |

Table 1t. Limited Phi-series, periods T, S, Distance R, Velocity Vi (Inverse) \& Vr (ref. unity).
All of which are readily available for real-time testing augmented by the occurrence of Phi as the mean period of revolution ( $T$ ) of Mars and the mean orbital velocity ( $V r$ ) of Mercury by way of the following Phi-series relations:

$$
\begin{align*}
& \text { Orbital velocity of Mercury } \operatorname{Vr} f(t)=\text { Period of Mars } T f(t) \text {, mean: }  \tag{23}\\
& \text { Mercury-Venus Syn SC7-8 S } f(t)=\frac{\text { Venus } f(t) \cdot \operatorname{Mercury} f(t)}{\operatorname{Venus} f(t)-\operatorname{Mercury} f(t)} \text {, mean: }{ }^{-2}(0.38196601125)  \tag{24}\\
& \text { Mercury, Heliocentric distance } R f(t) \text { from Mercury } f(t) \text {, mean: }{ }^{-2}(0.38196601125) \tag{25}
\end{align*}
$$

The associated parameters can be extended in range and complexity by the inclusion of Mars and and Earth, e.g., over the four years extending from 2045 to 2049 CE:


Fig 3a. Six-hourly data for Mars, Earth, Venus and Mercury, 2045-2049 CE ( $k_{1}=, k_{2}={ }^{0}=$ Unity (Earth, $\left.T, R, V r\right)$.

Figure 3a I- II waveforms are primarily concerned with Phi equated with (1): the period of revolution (T) of Mars, (2): the relative velocity (Vr) of Mercury, (3): the varying distance ( $R$ ) of Mars, (4): the Venus / Earth synodic ( $S$ ), and lastly, (5): the Earth $(T)$ / Venus ( $T$ ) ratio. In addition to the various intersections, the generally balanced position of the Mercury velocity $(V r)$ about the central constant (Phi) is of interest, as is the periodic grazing of the latter by the perihelion periods/positions of Mars. Also apparent for the longer waveforms are the maxima and minima around the end of 2046 and/or the beginning of the year 2047.

Figure 4b III. In order to examine the Venus waveforms in the same manner a different vertical scale was required. In addition to the standard waveform for Venus $T f(t)$, the ratios below were selected because they provide close, unequivocal maximum and minimum dates for the four-year test interval. Also, one of the chosen ratios utilising unity ( ${ }^{\circ}=1$ instead of the real-time periods of Earth about this mean) provided pheidian intercepts with the Venus waveform as follows:


Fig. 4b. Six-hourly data for Venus and Earth, 2045 to 2049. $k_{1}={ }^{1}=0.61803398875 .\left(k_{2}={ }^{0}=\right.$ Unity, Earth, $\left.T, R, V r\right)$.
In fact, the dates are similar to the maxima and minima around the middle of the present century seen in Figures $4 a, 4 b$ and 5 concerned with the motions of Jupiter and Saturn. It remains now to include the motions of Uranus and Neptune over the same initial interval (2000-2100 CE).

## Waveforms for Saturn-SD2-Neptune, plus the outermost triple Uranus-SD3-Neptune.

The real-time planetary data for Saturn, SD2 and Uranus from 2000 to 2100 CE with $k=1.61803398875$ resulted in only two pheidian intercepts, once again around the middle of the present century (Figure 5). Also, as in the cases for Jupiter and Saturn over the same period the waveforms were similarly symmetrical. Whereas, in keeping with increasing departures from the Phi-series periods beyond Uranus and the first resonant triple for the outermost pair of planets supplied by the Peirce Divisors [ 1, 1, 2] no intercepts occur for the Uranus : SD3 : Neptune trio.


Fig. 5. Daily real-time period ratios \& roots for Saturn, SD2 and Uranus from 2000 to 2100 CE, $k==1.61803398875$.
The associated time-series waveforms for the same formulas do, however, provide both phase and amplitude. As such the latter are included with the Jupiter-Saturn and Saturn-Uranus data in the last two intervals, i.e., Figure 6a2 from 2000 to 2100 CE and Figure 6e from 2415 to 2515 CE.


Fig 6a. Daily period ratios: Jupiter T, SD1, Saturn $T$, SD2 and Uranus $T$ from 2000 to 2100 CE, $k==1.61803398875$.


Fig 6b. Daily period ratios: Jupiter $T, S D 1$, Saturn $T, S D 2$ and Uranus $T$ from 1900 to $2000 C E, k==1.61803398875$.


Fig 6c. Daily period ratios: Jupiter T, SD1, Saturn T, SD2 and Uranus $T$ from 1800 to 1900 CE, $k==1.61803398875$.


Fig 6d. Daily period ratios: Jupiter $T, S D 1$, Saturn $T, S D 2$ and Uranus $T$ from 1700 to 1800 CE, $k==1.61803398875$.


Fig. 6a2. Daily period ratios \& roots: Jupiter, SD1, Saturn, SD2, Uranus, SD3 \& Neptune, 1800-1900 CE, $k==1.61803398875$.


Dates for Figure 6 e are adjusted for comparison with the waveforms of Figure 6a2. No earlier/similar adjustments
are possible since the generating power-series data commences on January 1, 1600 CE. Nonetheless, it seems more complex intersections about the constant occur during this earlier period, especially near the end points of the interval 1640 to 1695 . Further investigation reveals a similar occurrence to the former circa 2542-2557 CE, but at this point, remaining with Figures 6a through 6e, it is uncertain to what extent these waveforms provide sufficient information to confirm Kepler's interest ${ }^{6}$ in an 800-year cycle between Jupiter and Saturn. Or, from the tests here, an approximate 795 -year cycle with a possible maximum before the middle of the present century, i.e., December 2046. Then again, there are the Jupiter-Saturn grand alignments that occur some 25 years earlier than the dates of interest (March 1226, July 1623 and December 2020). Then again, 795-year maxima before 2047 (1252, 457, -338, $-1133,-1928,2723,-3518,-4313,-5108,-5903$ ) and/or the 397.5 -year half-cycles are not necessarily of immediate historical significance in terms of cyclic causes-and-effects, etc. The current concerns with global warming on the other hand may not lie so much in such cycles per se, but the troubling possibility that present industrial activity could conceivably tip the balance beyond the "norm" with potentially disastrous consequences.

In any event, such theorising lies well beyond the present examination of the consecutive Pheidian constants ${ }^{-3},-2,{ }^{-1}, 0,{ }^{1},{ }^{2}$ and the preliminary real-time survey of the planetary waveforms of the Superior and Inferior planets tentatively explored here.

As for the generation of the data applied in Figures 1 through 6, it could be suggested that methods developed by Bretagnon and Simon are truly excellent, but however intriguing, that the results are nonetheless still subject to a potentially troublesome Korzibskian qualifier, namely, that:

> "The Map is not the Territory."

Nevertheless, whether embedded in the methodology, the Solar System, or indeed both, the Golden Ratio is to a certain degree undoubtedly present, enough at least, to justify Benjamin Peirce's research and his conclusions.

This said, and bearing in mind the observation made earlier concerning the reduction of the base period(s) for HR 8799 by Pierce's limiting constant ${ }^{-2}$, a matter of considerable import remains.

## The Peirce planetary framework and Phyllotaxis

In addition to the limiting value of Pierce's planet-to-planet reduction ratios, the constant ${ }^{-2}$ ( 0.38196601125 ) has long been associated with phyllotaxis and the ideal growth angle of 137.50776 degrees obtained from a variety of viewpoints that include revolution-associated multiplication or division. Thus either from $360^{-2}$ or the reciprocal constant $\left.2.61803398875(360 /)^{2}\right)^{7}$ and also the more convenient Fibonacci ratio 55/144 ( $0.3819444^{*}$ ) =137.5/360 .

What do phyllotaxis and the natural growth angle have to do with the present discourse ?
Firstly, the title of Benjamin Pierce's initial paper ${ }^{8}$ was "Mathematical Investigations of the Fractions Which Occur in in Phyllotaxis." Secondly, his hypothesis in the present astronomical context was precisely summarized in terms of phyllotaxis by Louis Agassiz in the Essay on Classification, the entire set of the planet-to-planet Fibonacci reduction ratios included. Thus Agassiz states: ${ }^{9}$

> It is well known that the arrangement of the leaves in plants may be expressed by very simple series of fractions, all of which are gradual approximations to, or the natural means between $1 / 2$ or $1 / 3$, which two fractions are themselves the the maximum and the minimum divergence between two single successive leaves. The normal series of fractions which expresses the various combinations most frequently observed among the leaves of plants is as follows: $1 / 2,1 / 3,2 / 5,3 / 8$, $5 / 13,8 / 21,13 / 34,21 / 55$, etc. Now upon comparing this arrangement of the leaves in plants with the revolutions of the members of our Solar System, Peirce has discovered the most perfect identity between the fundamental laws which regulate both. (italics supplied).

There are a number of reasons why this quotation is given here rather than in the Introduction. The first stems from the negative reception of Benjamin Pierce's hypothesis and the manner that it was cast aside. The second reason is related to the first in so much as it is almost certain that the synodic component would have been incorporated far earlier had the work received a favorable reception. Here readers may decide for themselves. The third reason is more pragmatic; even with the inclusion of the Phi-series it was felt that refinements to Pierce's planetary model might still be required before embarking on standard tests beyond the Solar System.

This said, a short introduction and an acknowledgment of both earlier and recent contributions appears to be in order, beginning with the "three-fold number," now that readers are acquainted with both the number and the significance assigned to the matter by Benjamin Pierce and Louis Agassiz.

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## PART FOUR



THE FIBONACCI SERIES, THE LUCAS SERIES
AND PLATONIC TRIANGLES

## HISTORICAL EXTENSIONS AND FURTHER IMPLICATIONS

## Historical underpinnings

Commenting on Aristotle's De caelo et mundo,' medieval French scholar Nicole Oresme (1320-1382 CE) sought fit to include the cryptic statement "All Things are Three," augmented in part by the following quotation by Ovid: ${ }^{2}$

> Said Aristotle, lord and prince of Greek philosophers and never-failing friend of Truth: All Things are three; the three-fold number is present in all things whatsoever, nor did we ourselves discover this number, but rather, nature discovers it for us.

Antiquity and bald statement notwithstanding, the identity of the "three-fold number" and reference to "nature" in the above are abundantly clear. Indeed, since Oresme's time the latter has been variously named (Divine Ratio, Divine Section, Golden Section, Golden Ratio, etc.) and also discussed at length, earlier observations by Fibonacci (ca.1127-1240 CE) and series of the same name included. But, as noted earlier, it is unlikely that this intriguing series would have escaped the attention of early Greek philosophers or other inquiring minds. Nonetheless, for continuity the "Fibonacci series" has been retained as such throughout the present work, with the "three-fold number," and/or Golden Ratio defined here as the limiting value of the ratios of adjacent numbers (the larger over the lesser) of the Fibonacci series. The latter, of course, already discussed in terms of reduction ratios selected by Benjamin Pierce having now come full circle to re-embrace phyllotaxis and natural growth.

Even so, well before this the quest for enlightenment appears to have been enduring, and to some extent at least, successful. Johannes Kepler (1571-1630CE), for example, following Aristotle's use of "planes," ${ }^{3}$ introduced the series " $1,1,2,3,5,8,13,21$ " and associated ratios before declaring: ${ }^{4}$ "in the flower is displayed a pentagonal standard, so to speak." Later, N. Grew ${ }^{5}$ surmised that "from the contemplation of Plants, men might first be invited to Mathematical Enquirys," while more widely and more recently modern mathematician lan Stewart observed in Nature's Numbers (1995) that "nature leaves clues for the mathematical detectives to puzzle over." ${ }^{6}$

Historically, following Fibonacci inquires concerning the mathematical complexities of phyllotaxis and related matters have extended from the dimensions and forms of shells investigated by Canon Mosely ${ }^{7}(1838)$ to the very structure of the Solar System pursued in the $19^{\text {th }}$ Century by Benjamin Peirce. ${ }^{8}$ The general inquiry was continued by Arthur H. Church ${ }^{9}$ (1904), Sir Theodore A. Cook ${ }^{10}$ (1914), Sir D'arcy Wentworth Thompson ${ }^{11}$ (1917), R. C. Archibald ${ }^{12}$ (1919), Samuel Colman ${ }^{13,14}$ (1920) and Jay Hambidge ${ }^{15}$ (1920) in the early part of the previous century on into the next. And significantly, the recent revelation (2021) by A. Asadi ${ }^{16}$ that "the constant 1.618 can be seen everywhere" in the remains of the Apadama Palace in the ancient Persian city of Persepolis (ca. 550 BCE).
Lastly, to this partial list can also be added since the start of the $21^{\text {st }}$ Century related material made available on on the World-Wide Web by scientists, academic institutions and interested parties.

## Historical extensions

As far as Benjamin Pierce's unappreciated contribution and the stagnation that followed are concerned it is most unfortunate that the synodic component and the natural expansions were not addressed in his time. Sadly, in view of the swift rejection of his research, it is also understandable now why Benjamin Pierce in his capacity as President of The American Association for the Advancement of Science stressed freedom from being "helplessly exposed to the assaults of envious mediocrity," in his otherwise inspirational outgoing speech to the Association in 1853. ${ }^{17}$ This said, the contents of the speech with its numerous erudite references to the past suggest, even without synodic components, that his understanding of the matter in both time and place was extensive. Indeed, it becomes quite apparent that the information pertaining to this matter in ancient writings is detailed, complex, comprehensive and as yet to fully enter the mainstream of modern science. Part of the difficulty lies in the oddly neglected subject of orbital velocity, particularly as treated in Plato's well-known dialogue the Timaeus, as Galileo (1638) ${ }^{18}$ and more recently Harris (1989) ${ }^{19}$ unsuccessfully attempted to impart. Why this deficiency still persists is hard to understand, but with the inclusion of velocity as applied in Phi-series Table 3 it can be shown that humankind has not lost as much ancient knowledge as currently thought. In fact, the underlying core seems to have survived, preserved, as Thomas Taylor (1785-1837) notes, by way of the teachings of Orpheus, Plato and Pythagoras, the first: "mystically and symbolically; by the second, enigmatically, and through images; and scientifically by the third." ${ }^{20}$ But even so, a considerable amount of material remains to be analyzed and if anything there is too much with too little guidance. Fortunately, initial help and a narrower focus are supplied by the Neo-platonist Proclus (410-485 CE), who wrote: ${ }^{21}$

IfI had it in my power, out of all the ancient books I would suffer to be current only the Oracles ${ }^{22}$ and the Timaeus. ${ }^{23}$
The latter-one of Plato's better known dialogues-is the more familiar, whereas the former still remains relatively obscure, if not entirely arcane. This is, however, the same Proclus who begins his own commentary on the Timaeus
with an introduction which is remarkable for its scope and caustic certitude; to wit: ${ }^{21}$
That the design of the Platonic Timaeus embraces the whole of physiology, and that it pertains to the theory of the universe, discussing this from the beginning to the end, appears to me to be clearly evident to those who are not entirely illiterate.

At which point temporal racism - that this (whatever) could not be known by (whomever) in those days (whenever) likely cuts in with most modern readers. But in the final analysis it still comes down to the details and overall picture which emerges, assuming such a progression does indeed prove to be feasible.

This said, at least parts of the Chaldean Oracles are relatively straightforward, whereas mathematical components in Plato's Timeaus and related sources generally remain unclear, despite detailed examination by modern scholars, e.g., de Santillana (1969), ${ }^{24}$ Cornford (1975), ${ }^{25,26}$ Brumbaugh (1977), ${ }^{27}$ McClain (1978), ${ }^{28}$ and two compendia on the "The Harmony of the Spheres" assembled by Godwin (1993). ${ }^{29,30}$ Cornford, however, supplies a further qualifier, for he warns that: "the Timaeus covers an immense field at the expense of compressing the thought into the smallest space. ${ }^{31}$ More specifically, the two best known numerical transformations in the Timaeus concern firstly the role played in the construction of the "World-Soul" by the Double [1, $2,4,8$ ] and the Triple [1,3,9,27] intervals. The latter pair have already been discussed briefly with respect to the velocity components of the laws of planetary motion, but further investigations reveal that there is indeed far more to the Timaeus, including a return to the Pierce divisors and the associated planetary framework as explained next.

## The Platonic Triangles, the rotation of the elements, and the Music of the Spheres.

Among the better-known mathematical conundrums in the Timaeus is the emphasis placed on two specific types of triangles, namely the isosceles and the equilateral which are described as follows: ${ }^{32}$


#### Abstract

Now, of the two triangles, the isosceles is of one type only; the scalene, of an endless number. Of this unlimited multitude we must choose the best, if we are to make a beginning on our principles. For ourselves, however, we we postulate as the best of these many triangles one kind, passing over all the rest; that, namely, a pair of which compose the equilateral triangle ... the one isosceles (the half-square), the other having the great side triple in square of the lesser (the half-equilateral)...[54b.] We must now be more precise upon a point that was not clearly enough stated earlier. It appeared as though all the four kinds could pass through one another into one another; but this appearance is delusive; for the triangles [54c.] we selected give rise to four types, and whereas three are constructed out of the triangle with unequal sides, the fourth alone is constructed out of the isosceles. Hence it is not possible for all of them to pass into one another by resolution, many of the small forming a few of the greater and vice versa. But three of them can do this; for these are all composed on one triangle, and when the larger bodies are broken up several small ones will be formed of the same triangles, taking on their proper figures; and again when several of the smaller bodies are dispersed into their triangles, the total [54d.] number made up by them will produce a single new figure of larger size, belonging to a single body. So much for their passing into one another. (Timaeus, 53d-54d, translation by Francis MacDonald Cornford).


It remains to show that the isosceles triangle not only pertains to the Fibonacci series but also the Pierce ratios in their resonant triple form. Furthermore, in a like manner it can be demonstrated that the equilateral triangle produces the Lucas series, and here once again in resonant form.

Either way, insights concerning this material are provided by a Babylonian mathematical tablet (YBC 7289) from the Old Babylonian period in the shape of an "ellipsoid" that circumscribes a square with both the diagonals and value for the side of the square obtained from the Babylonian estimate for root of two: 1;24,51,10 (1.4142129). ${ }^{33}$ Thus the diagram has four isosceles triangles with the hypotenuse obtained from the radius and the Pythagorean theorem some 1500 hundred years before the time of Pythagoras. This is nothing new, and nor apparently, is the application in the present context, perhaps best described as "The Rotation of the Elements," as described by John Opsopaus (1995), ${ }^{34}$ who included the following line from Alchemist George Ripley (d.1490): "When thou hast made the quadrangle round, Then is all the secret found." Opsopaus adds later that "The rotation of the elements is a key alchemical procedure, the principal means by which the purified essence of a substance is extracted and raised to its most sublime state."

## The Isosceles Triangle, the Equilateral Triangle, and the Rotation of the Elements

As for the "rotations," they proceed in 90-degree stages, thus four "rotations" per revolution commencing with the parameters of the two specified right triangles. Initial values for the isosceles triangle are Base =1, Perpendicular=1, Hypotenuse $=\checkmark 2$. For the first $90^{\circ}$ rotation the initial base of 1 is retained but the Perpendicular (1) is replaced by the original hypotenuse $(\checkmark 2)$ with the new hypotenuse $(\checkmark 3)$ obtained from the Pythagorean theorem, and so on. Shown in Table 1, the successive squares of the sides of the first triangle are in due order immediately recognizable as the resonant Fibonacci Triples (RZT) in company with the Peirce period divisors en route to the limiting triangle:

| GROWTH/90 ${ }^{\circ}$ ( $1 / 4$ Rotations) | ISOSCELES Sides: B, P \& H | EXPANDING <br> Rt. Triangles | SIDES ${ }^{2}$ <br> Fib.Triple | B. PIERCE Reduction | RATIO OF SIDES P/B, H/B (Ref B) | ANGLES ( $\angle$ ) Degrees ( $\mathrm{B}_{60}$ ) |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Initial Triangle | BASE B1 = 1 | 1 | 1 | NEPTUNE | 1 | Initial Angle |
| Timaeus (54a) | PERP. P1 = 1 | 1 | 1 | 1/2 | 1 | (ISOSCELES) |
| $\mathrm{H}=\checkmark\left(\mathrm{B}^{2}+\mathrm{P}^{2}\right)$ | HYP. $\mathrm{H} 1=\sqrt{ } 2$ | 1.4142136 | 2 | Uranus | 1.41421356237 | $45^{\circ}$ |
| Quadrant \#1 | BASE B2 = P1 | 1 | 1 |  | 1 |  |
| $1 / 4$ Cycle | PERP P2 = H 1 | 1.4142136 | 2 | 1/3 | 1.41421356237 |  |
| $90^{\circ}$ | HYP H2 (calc.) | 1.7320508 | 3 | Saturn | 1.73205080757 | 54;44,8,12 |
| Quadrant \#2 | BASE B3 = P2 | 1.4142136 | 2 |  | 1 |  |
| $1 / 2$ Cycle | PERP P3 = H2 | 1.7320508 | 3 | 2/5 | 1.22474487139 |  |
| $180^{\circ}$ | HYP H3 (calc.) | 2.2360679 | 5 | Jupiter | 1.58113883008 | 50;46,6,33 |
| Quadrant \#3 | BASE B4 = P3 | 1.7320508 | 3 |  | 1 |  |
| $3 / 4$ Cycle | PERP P4 = H3 | 2.2360679 | 5 | 3/8 | 1.29099444874 |  |
| $270{ }^{\circ}$ | HYP H4 (calc.) | 2.8284271 | 8 | (M-J Gap) | 1.63299316186 | 52;14,19,30 |
| Quadrant \#4 | BASE B5 = P4 | 2.2360679 | 5 |  | 1 |  |
| Revolution \#1 | PERP P5 = H4 | 2.8284271 | 8 | 5/13 | 1.26491106407 |  |
| $360{ }^{\circ}$ | HYP H5 (calc.) | 3.6055513 | 13 | Mars | 1.61245154966 | 51;40,16,15 |
| Quadrant \#5 | BASE B6 = P5 | 2.8284271 | 8 |  | 1 |  |
| $1 / 4$ Cycle | PERP P6 = H5 | 3.6055513 | 13 | 8/21 | 1.27475487839 |  |
| $450{ }^{\circ}$ | HYP H6 (calc.) | 4.5825757 | 21 | Venus | 1.62018517460 | 51;53,13,28 |
| Quadrant \#6 | BASE B7 = P6 | 3.6055513 | 13 |  | 1 |  |
| $1 / 2$ Cycle | PERP P7 = H6 | 4.5825757 | 21 | 13/34 | 1.27097781860 |  |
| $540{ }^{\circ}$ | HYP H7 (calc.) | 5.8309519 | 34 | Mercury | 1.61721508013 | 51;48,16,8 |
| Quadrant \#7 | BASE B8 = P7 | 4.5825757 | 21 |  | 1 |  |
| $3 / 4$ Cycle | PERP P8 = H7 | 5.8309519 | 34 | (21/55) | 1.27241802057 |  |
| $630^{\circ}$ | HYP H8 (calc.) | 7.4161985 | 55 | (IMO \#1) | 1.61834718743 | 51;50,9,38 |
| Quadrant \#8 | BASE B9 = P8 | 5.8309519 | 34 |  | 1 |  |
| Revolution \#2 | PERP P9 = H8 | 7.4161985 | 55 | (34/89) | 1.27186754767 |  |
| $720^{\circ}$ | HYP H9 (calc.) | 9.4339811 | 89 | (IMO \#2) | 1.61791441641 | 51;49,26,16 |
| \{ Quadrants 9 through 27 omitted \} |  |  |  |  |  |  |
| Quadrant \#28 | BASE B29 $=$ P28 | 717.09762 | 514229 | $\mathrm{B} 29=1$ | 1 | Limiting |
| Revolution \#7 | PERP P29 $=$ H28 | 912.16227 | 832040 | P29 $\approx \checkmark$ | 1.27201964951 | Angle: |
| $2520^{\circ}$ | HYP H29 (calc.) | 1160.2883 | 1346269 | $\mathrm{H} 29 \approx$ | 1.61803398875 | 51;49,38,15 |

Table 1. The Isosceles triangle, the "Rotation of the Elements" I and the Pierce reduction ratios.
Limits, with angles given to the third sexagesimal place, dimensions of the fourth dual triangle and growth per quadrant/per revolution are as follows:

LIMIT: Base angle: 51;49,38,15. ${ }^{\circ}$ Vertex: 38;10,21,45. ${ }^{\circ}$ LIMIT: Proportion: $1,,^{1},^{1}$ (The Half-Phi-Series). Double vertex: 76;20,43,30. Double Triangle: Base 2, Height 1.2720196495, both sides 1.61803398875.

Growth per quadrant: $\checkmark$ (1.2720196495). Growth per Revolution: $(w)={ }^{2}$ (2.61803398875), Relation (8).


THE ISOSCELES TRIANGLE FULL/HALF EQUILATERAL TRIANGLES


ROTATION OF THE ELEMENTS (Timaeus, 54a)

## The Half-equilateral triangle and the Rotation of the Elements

Even so, there is far more to this approach, for the same procedure commencing with the half-equilateral triangle uses the Half Phi-series as opposed to the Phi-series introduced earlier in Table 3. Here, however, "rotations" proper commence at Mars and move outwards towards the Half Phi-series as shown for the twelve quadrants and three revolutions between the latter planet and Saturn in Table 2. Included here, albeit as pheidian exponents, are the "Fourth," (4:3), "Fifth" (3:2), "Major Six" (5:3) and "Octave" (2:1) with the Fourth and the Octave also two primary constants for the increases in planetary distances and periods of revolution respectively, thus Phi-series relations (9) and (7).

| GROWTH/90 ${ }^{\circ}$ ( $1 / 4$ Rotations) | 1/2 EQUILATERAL Sides: B, P \& H | SIDES ${ }^{2}$ <br> Lucas RZT | LUCAS-BASED Right triangles | HALF -SERIES Periods (Years) | ½ T PLANORBIDAE |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  |  |  |  | Exp. | $\mathrm{T}^{1 / 3}=$ Velocity ( |  |
| Initial Triangle | BASE B1 $=1$ | 1 | 1 | (1) | (0) | PHEIDIAN SIXT |  |
| Rev. 0 MARS | PERP. P1= $\sqrt{ } 3$ | 3 | 1.7320508076 | 1.6180339887 | 1 | Exponents: |  |
| $H=\checkmark\left(B^{2}+P^{2}\right)$ | HYP. H1 $=2$ | 4 | 2 | 2.0581710273 | 1.5 | [1.083505882] | 1 |
| Quadrant \#1 | BASE B2 $=$ P1 | 3 | 1.7320508076 | 1.6180339887 | 1 | 1.1739849967 | 2 |
| $1 / 4$ cycle | PERP P2 = H1 | 4 | 2 | 2.0581710273 | 1.5 | 1.2720196495 | 3 |
| $90^{\circ}$ | HYP H2 (calc.) | 7 | 2.6457513111 | 2.6180339887 | 2 |  |  |
| Quadrant \#2 | BASE B3 $=$ P2 | 4 | 2 | 2.0581710273 | 1.5 |  |  |
| $1 / 2$ cycle (Syn.) | PERP P3 = H2 | 7 | 2.6457513111 | 2.6180339887 | 2 | 1.3782407725 | 4 |
| $180^{\circ}$ | HYP H3 (calc.) | 11 | 3.3166247904 | 3.3301906768 | 2.5 | $(4 / 6$ Relation 10) |  |
| Quadrant \#3 | BASE B4 = P3 | 7 | 2.6457513111 | 2.6180339887 | 2 |  |  |
| $3 / 4$ cycle | PERP P4 = H3 | 11 | 3.3166247904 | 3.3301906768 | 2.5 | 1.4933319840 | 5 |
| $270{ }^{\circ}$ | HYP H4 (calc.) | 18 | 4.2426406871 | 4.2360679775 | 3 |  |  |
| Quadrant \#4 | BASE B5 = P4 | 11 | 3.3166247904 | 3.3301906768 | 2.5 |  |  |
| Rev. 1 (MJ-Gap) | PERP P5 = H4 | 18 | 4.2426406871 | 4.2360679775 | 3 | 1.6180339887 | 6 |
| $360{ }^{\circ}$ | HYP H5 (calc.) | 29 | 5.3851648071 | 5.3883617041 | 3.5 | ( ${ }^{6 / 6}$ Relation 8) |  |
| Quadrant \#5 | BASE B6 = P5 | 18 | 4.2426406871 | 4.2360679775 | 3 |  |  |
| $1 / 4$ Cycle | PERP P6 = H5 | 29 | 5.3851648071 | 5.3883617041 | 3.5 | 1.7531493444 | 7 |
| $450{ }^{\circ}$ | HYP H6 (calc.) | 47 | 6.8556546004 | 6.8541019663 | 4 |  |  |
| Quadrant \#6 | BASE B7 = P6 | 29 | 5.3851648071 | 5.3883617041 | 3.5 | Fourth 4:3 (8/6) |  |
| $1 / 2$ cycle (Syn.) | PERP P7 = H6 | 47 | 6.8556546004 | 6.8541019663 | 4 | 1.8995476269 | 8 |
| $540{ }^{\circ}$ | HYP H7 (calc.) | 76 | 8.7177978871 | 8.7185523808 | 4.5 | ( ${ }^{4 / 3}$ Relation 9) |  |
| Quadrant \#7 | BASE B8 = P7 | 47 | 6.8556546004 | 6.8541019663 | 4 | Fifth 3:2 (9/6) |  |
| $3 / 4$ cycle | PERP P8 = H7 | 76 | 8.7177978871 | 8.7185523808 | 4.5 | 2.0581710273 | 9 |
| $630^{\circ}$ | HYP H8 (calc.) | 123 | 11.090536506 | 11.090169944 | 5 |  |  |
| Quadrant \#8 | BASE B9 = P8 | 76 | 8.7177978871 | 8.7185523808 | 4.5 | Major Six 5:3 (10 |  |
| Rev. 2 JUPITER | PERP P9 = H8 | 123 | 11.090536506 | 11.090169944 | 5 | 2.2300404146 | 10 |
| $720^{\circ}$ | HYP H9 (calc.) | 199 | 14.106735980 | 14.106914085 | 5.5 | ( ${ }^{10 / 6}$ Jupiter Vi) |  |
| Quadrant \#9 | BASE B10 = P9 | 123 | 11.090536506 | 11.090169944 | 5 |  |  |
| $1 / 4$ cycle | PERP P10 = H9 | 199 | 14.106735980 | 14.106914085 | 5.5 | 2.4162619067 | 11 |
| $810^{\circ}$ | HYP H9 (calc.) | 322 | 17.944358445 | 17.944271910 | 6 |  |  |
| Quadrant \#10 | BASE B11 = P10 | 199 | 14.106735980 | 14.106914085 | 5.5 | Octave 2:1 (12/6 |  |
| 1⁄2 cycle (Syn.) | PERP P11 = H10 | 322 | 17.944358445 | 17.944271910 | 6 | 2.6180339887 | 12 |
| $900{ }^{\circ}$ | HYP H11 (calc.) | 521 | 22.825424421 | 22.825466466 | 6.5 | (12/6 Relation 7) |  |
| Quadrant \#11 | BASE B12 = P11 | 322 | 17.944358445 | 17.944271910 | 6 |  |  |
| $3 / 4$ cycle | PERP P12 = H11 | 521 | 22.825424421 | 22.825466466 | 6.5 | 2.8366552265 | 13 |
| $990^{\circ}$ | HYP H12 (calc.) | 843 | 29.034462282 | 29.034441854 | 7 |  |  |
| Quadrant \#12 | BASE B13 $=$ P12 | 521 | 22.825424421 | 22.825466466 | 6.5 |  |  |
| Rev. 3 SATURN | PERP P13 $=\mathrm{H} 12$ | 843 | 29.034462282 | 29.034441854 | 7 | 3.0735326237 | 14 |
| $1080{ }^{\circ}$ | HYP H13 (calc.) | 1364 | 36.932370625 | 36.932380550 | 7.5 | $\left({ }^{1 / 6}\right.$ Saturn Vi) |  |

Table 2. "Rotation of the Elements" II. Lucas and Half-Phi series, the Fourth, the Fifth, the Major Six and Octave. The Sixths (Vi) are color-coded growth rates ( $w$ per revolution) for Pheidian test spirals ( $w={ }^{1 / 6}$ to ${ }^{146)}$ ) from Mars to Saturn. The above range brackets the majority of equi-angular spirals found among ammonites and numerous (but not all) shells.

Thus, while emphasizing the Fibonacci series the first set generates the Pierce reduction ratios and the associated resonant triples that underly the present survey of the Solar System and Systems further afield. Above all else, the rotations pertain to the divisors for the periods of revolution and synodic cycles commencing with the outermost as later adopted by Pierce.

The half-equilateral triangle on the other hand, although again understood in terms of periods of revolution and synodic cycles involves the Lucas series, begins at Mars and proceeds outwards towards the Half-Phi-series with correspondence increasing with distance. However, there is now something else to consider, for the addition of quarter-cycle periods generates a rectangular ("square") spiral with a quadrantal growth rate of $\checkmark$ (1.2720196495 and ${ }^{2}$ (2.61803398875) per revolution, thus relation (8), the Phi-series planet-to-planet increment for the periods of revolution. The inclusion of the pheidian exponents $2 / 1,5 / 3,3 / 2$ and $4 / 3$ for $V i$ follows from these periods, but whether this is the "Harmony of the Spheres" per se is far from secure, general correspondence notwithstanding.

## Further considerations

It must be acknowledged that it helps to return to this material with details of the Pierce Divisor framework already in place. And also, of the two analyses presented here the latter is perhaps the more controversial. Even so, neither is entirely out of place, especially in light of the material analyzed by Jöran Friberg in A REMARKABLE COLLECTION of BABYLONIAN MATHEMATICAL TEXTS published in 2007. ${ }^{35}$ Particularly relevant here is the combination of both Greek and Babylonian texts dealing with something akin to the "rotation of the elements" just discussed.

Given below is the "spiral" expansion of the isosceles triangle obtained from continued $90^{\circ}$ rotations of the same with the last hypotenuse the base of the next, and so on. Rotated here, the figure is otherwise as given in Friberg's analysis of Old Babylonian mathematical text MCL 7028 originally treated by Neugebauer and Sachs ${ }^{36}$ as a table of logarithms and and exponents. Friberg's analysis is quite different, dealing with a "spiral chain algorithm," and the fixed rotation of the isosceles triangle as shown. Which in itself is not only of conceptual interest, but also a helpful exercise for the somewhat more complex rotations involving the Pythagorean expansions of the sides.

On the other hand, rotating the Fibonacci sides in Table 1 generates a tighter line spiral, which in turn leads to an a near equi-angular spiral with a growth rate per revolution ( $w$ ) of approximately 2.666:1, and similarly 2.765:1 from Lucas data rotations. Neither of the two rectangular expansions quite matches the equi-angular spiral based on ${ }^{2}$ (2.61803398875), but they are nonetheless still close, the Fibonacci variant especially.

Not to scale, the fixed format line spiral, the latter pair plus the Phi-series spiral based on the last constant ( ${ }^{2}$ ) with its uniform expansion rate of ${ }^{1 / 2}\left(1.27201964951\right.$ : 1) per $\left(90^{\circ}\right)$ quadrant are:


Fig. 1. (a) Rotations of Fixed Isosceles triangles, (b) Lucas rotations, (c) Fibonacci rotations, and (d) the Phi-series.

But what is the underlying purpose of such extensions, and how might they relate to obscure statements such as that already noted, namely Alchemist George Ripley's "When thou hast made the quadrangle round, Then is all the secret found." ? Perhaps - despite their arcane nature - further obscure explanations in later alchemical works, e.g., section 83 in the Hermetic Arcanum (1623) may be of assistance since this matter appears to involve two spirals in complex association with three circles. With, apparently, such "Circulations" considered to be "Nature's instruments, whereby the elements are prepared." Thus in full, section \#83 (of 138) from the Hermetic Arcanum: ${ }^{37}$
83. The Circulation of the Elements is performed by a double Whorl, by the greater or extended and the less or contracted. The Whorl extended fixeth all the Elements of the Earth, and its circle is not finished unless the work of Sulphur be perfected. The revolution of the minor Whorl is terminated by the extraction and preparation of every Element. Now in this Whorl there are three Circles placed, which always and variously move the Matter, by an Erratic and Intricate Motion, and do often (seven times at least) drive about every Element, in order succeeding one another, and so agreeable, that if one shall be wanting the labour of the rest is made void. These Circulations are Nature's Instruments, whereby the Elements are prepared. Let the Philosopher therefore consider the progress of Nature in the Physical Tract, more fully described for this very end.
Although still difficult, the above quotation provides some understanding of the materials already assembled here, particularly the two "whorls" (spirals), with the "greater or extended" reasonably the ever-extending Phi-series spiral ( $w==^{2}$ ), with the "lesser" confined, perhaps, to Solar System equivalents, or Fibonacci variants of the same.
The "three Circles" on the other hand, are simpler and less controversial in so much as the previous analyses of the real-time motions of the Jupiter, Synodic SD1 and Saturn included - although not shown - plan views of both the real-time orbits and mean circular orbits of Jupiter and Saturn with the "orbit" of SD1 necessarily in between. Hence three circles in a readily recognized configuration. For this purpose, however, the same mean sidereal periods used earlier for the mean synodic arcs are preferred since there is no ambiguity concerning their values.

As presented in ACT (1955) by Neugebauer, Babylonian mean periods were obtained from the following "simple" integer period relationships: ${ }^{38}$

Superior planets: N years : II Synodic arcs/periods : Z Rotations
Inferior planets: $N$ years :II Synodic arcs/periods. (only)
which, with revolutions substituted for Neugebauer's rotations and periods of revolution (Z) added for Venus and Mercury, are:

| SATURN: | 265 years (N), 256 Synodic arcs (II), | 9 Revolutions (Z) |
| :---: | :---: | :---: |
| JUPITER: | 427 years (N), 391 Synodic arcs (II), | 36 Revolutions (Z) |
| MARS: | 284 years (N), 133 Synodic arcs (II), | 151 Revolutions (Z) |
| VENUS: | 1151 years (N), 720 Synodic arcs (II), | 1871 Revolutions (Z added) |
| MERCURY | Y: 46 years (N), 145 Synodic arcs (II), | 191 Revolutions (Z added) |

yielding by simple division mean periods of revolution which, by way of the Harmonic law give the corresponding mean distances for Saturn, SD1 and Jupiter shown in Figure 2A.

> SATURN: 265 years / 9 Revolutions $=11.86111^{*}(11 ; 51,40)$ years. $R=9.535532$
> (Added: Synodic SD1 [Relation 1$]=19.86220818$ years. $\quad$ " $R$ " $=7.334182)$
> JUPITER: 427 years $/ 36$ Revolutions $=29.4444^{*}(29 ; 26,40)$ years. $R=5.200961$

The mean distance orbits are comparable with a figure in a "babylonian-clay-tablet-with-geometrical-problems." ${ }^{39}$ Where, as seen in Figures 2B and 2C, an astronomical indicator is suggested, i.e., an off-set vertical center line in so much as Solar System planetary orbits with their small eccentricities do indeed resemble off-set circles when shown in planview. Then again, the text may perhaps refer to a more mundane application, though just what that might be is unclear as opposed to the present provisional suggestion.


As for the priority of Jupiter and Saturn in the Hermetic Arcanum, the final paragraph is "The Times of the Stone," in which, after mentioning Capricorn and Aquarius with respect to Saturn, and later also Scorpio and Sagittarius, the Arcanum states: "And thus the Philosophers' admirable offspring taketh its beginning in the Reign of Saturn, and its end and perfection in the Dominion of Jupiter." ${ }^{40}$ Thus, not only the relative motions of these two planets, but also specific locations, which brings to mind what was discussed in the Part I Excursus concerning Saturn and Jupiter at the junction points of their twin sectors associated with their lines of apsides, i.e., approximations for and ${ }^{2}$.

As for the present context, the "Rotation of the Elements" as suggested here can be at least represented by the following diagram:


Fig. 3. The Resonant Triples and the Isoceles / Half-Equilateral Right Triangles (Timaeus 53c-54d).
subject to corrections and/or further refinements, provided (perhaps), by modern "Alchemists," or perhaps not.

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TIME AND TIDE

## SPIRAL FORMS IN TIME AND PLACE

## The initial pheidian equiangular spirals

After the preceding section it becomes clear that the preservation of knowledge, albeit convoluted, obscure and difficult was nevertheless continued forward over the intervening centuries albeit with the degree of success yet to be fully determined. As for the emphasis on the spiral form, perhaps this is best examined in terms of inquiries into Nature in keeping with the main aspects of the matter provided earlier by Ovid concerning the three-fold number with, perhaps, understandable interest in the many spirals which abound in nature fueling an ongoing inquiry into the details, and (where possible) the mechanics involved.

Either way these results lead next to a theoretical base (hereafter the Pheidian planorbidae), namely the $\phi$-Series parameters $\boldsymbol{T}, \boldsymbol{R}$ and $\boldsymbol{V i}$ with initial emphasis on exponents 4, 5, 6 and 7 which generate distance relation (9), the mean sidereal period ( $T$ ) for Jupiter, Synodic cycle SD1 ( $S$ ), and the mean sidereal period ( $T$ ) of Saturn. Thereafter cube roots of these four periods yield the inverse velocities ( $\boldsymbol{V i}$ ) from exponents $\mathbf{4 / 3 , 5 / 3 , 6 / 3}$ and $\mathbf{7 / 3}$ (hence the title "Thirds") and the relative velocities Vr from their respective reciprocals.

| x | Planets/Syn | $\phi$-Series $T=\phi^{\mathrm{x}}$ | $\phi$-Series $R$ | Thirds | Sixths | $\phi$-Series Vi | $\phi$-Series Vr |
| :--- | :--- | :--- | :--- | :---: | :---: | :---: | :---: |
| 4 | Synodic (SD) | 6.8541019662 | 3.608281187 | $4 / 3$ | $8 / 6$ | 1.899547627 | 0.526441130 |
| 5 | JUPITER \#4 | 11.090169944 | 4.973080251 | $5 / 3$ | $10 / 6$ | 2.230040415 | 0.448422366 |
| 6 | Synodic (SD1) | 17.944271910 | 6.854101966 | $6 / 3$ | $12 / 6$ | 2.618033989 | 0.381966011 |
| 7 | SATURN \#3 | 29.034441854 | 9.446602789 | $7 / 3$ | $14 / 6$ | 3.073532624 | 0.325358511 |

Table 1. $\phi$-Series mean periods (years), mean distances (a.u.) \& mean velocities (Vi), (Vr) ref. unity.
The central value: $V i=\phi^{2}=2.61803398875$ already plays a major role in phyllotaxis and from ancient and modern sources provides a close approximation for the ratio of the mean synodic arcs for Saturn and Jupiter. Furthermore, it is also the reciprocal of the Pierce limiting constant $\phi^{-2}(0.38196601125)$, the relative velocity $(V r)$ of SD1 between Jupiter and Saturn, also the mean distance for Mercury ( $R$ ), and lastly, the period ( $S$ ) of Mercury-Venus Synodic 7-8.

Plus, earlier and independently, it must be acknowledged that the above constants and the distance variant: $\phi^{4 / 3}$ (1.899547627) were derived by K. P. Butusov (1997) ${ }^{34}$ as Planet Period and Planet Distance Laws which are essentially relations (8) and (9) as developed and applied here. The main difference being, it would seem, that the latter also integrated relation (1) but not as the general synodic formula initially applied here to complete Benjamin Pierce's stalled planetary framework.

## Spira Solaris

Thus the spiral Spira Solaris, $\mathrm{Vi}_{\mathrm{i}}=2.61803398875$, so-named because of its central importance above commencing with the equiangular "square" followed by the corresponding equiangular spiral which is inverted as shown below to provide a series of scalable spirals conforming to the free-swimming orientation of nautiloids and ammonites.


Fig.2a. The equiangular "square" spiral.


Fig.2b. Equiangular spiral "Spira Solaris."


Fig. 2c. "Spira Solaris" Inverted

## The Pheidias Spiral

Applied to an expansion factor provided by Phi itself ( $w=1.61803398875$ per revolution) the name Pheidias Spiral owes its origins to William Schooling's investigation of the mathematical relationships inherent in the Phi-series published in Cook's The Curves of Life (1914). ${ }^{36}$ Shown in Figure 3 in its original form the Pheidias spiral is inverted, augmented by inner whorls, then color-coded to standardize with Spira Solaris and subsequent test spirals.


William Schooling


The Pheidias Spiral Inverted with added whorls


Spira Solaris
TEST FORMAT 1


Spira Solaris
TEST FORMAT 2

Fig.3. The Pheidias Spiral and Spira Solaris equiangular Test spirals. Line: 1, Solid: 2. Test spirals are equal in size, not scale.

## Generation of equiangular spirals

At this juncture it should be noted that decades before the general availability of electronic computers a surprising number of investigations concerning growth, form and phyllotaxis were carried out during the first quarter of the previous century. It is, however, the wide-ranging Curves of Life (1914) by Sir Theodore Andreas Cook and even more expansive On Growth and Form (1917) by Sir d'Arcy Wentworth Thompson which are of immediate interest. It is the latter which provides the mathematical details concerning equiangular spirals, in particular those associated with The shape of a nautiloid spiral ${ }^{37}$ with attendant whorl-to-whorl growth ( $w$ ), the equiangles, and the following formula for the radius vectors:

$$
\begin{equation*}
r=e^{2 \pi \operatorname{Cot} \alpha} \tag{1}
\end{equation*}
$$

This was expanded by wider descriptions of the general form of the coiled shell in $1966^{38}$ and $1967{ }^{39}$ by David Raup (1933-2015) whereas the initial approach employed by Thompson included a table of corresponding growth factors ( $w=1 / r$ ) and another in the form $r: 1$ with corresponding equiangles, stating "Here we have $r=e^{2 \pi \operatorname{Cot} \alpha}$, or $\log r=\log e \times 2 \pi \times \cot \alpha$, from which we obtain the following figures," ${ }^{40}$ explaining in a footnote: "It is obvious that the ratios of opposite whorls, or of radii $180^{\circ}$ apart, are represented by the square roots of these values; and the ratios of whorls or radii $90^{\circ}$ apart, by the square roots of these again." Which returns the inquiry to the Rotation of the Elements, the equiangular "square" and additional insights from Aristotle's cryptic statement in On the Heavens, namely: ${ }^{41}$
..... It is agreed that there are only three plane figures which can fill a space, the triangle, the square and the hexagon, and only two solids, the pyramid and the cube.
One can certainly generate all that is needed from relation (1) and above information provided by Thompson for any equiangular spiral required, but this relation, involving e, logarithms, radians and trigonometric functions is cumbersome, and also, as it turns out, unnecessary. Identical results for $r$ per degree can in fact be obtained by the expansion of Aristotle's three figures, i.e., addition of an equiangular triangle ( $120^{\circ}$ ) to the equiangular square ( $90^{\circ}$ ) followed by an equiangular hexagon ( $60^{\circ}$ ), etc. At which point, extending the process downwards to include $45^{\circ}$, $30^{\circ}, 15^{\circ}, 36^{\circ}, 24^{\circ}, 18^{\circ}, 12^{\circ}, 6^{\circ}, 3^{\circ}, 2^{\circ}$ until, leading to an equiangular 360 -gon (triacosihexacontigon) all the "spaces" are filled, i.e., comprised of $1^{\circ}$ segments. Or, better stated, the straight-line bases of narrow triangles per degree sufficient to generate all the equiangular spirals shown here according to the desired expansion rates per revolution. Furthermore, because the growth factors are already known, equiangles play no role in the generation of the spirals, nor do $e, \pi$, logarithms or trigonometric functions.

What are required for test purposes are standard formats and ranges which originally pertained to Spira Solaris, Pheidias and immediate vicinity in an astronomical context. Thus mean periods of revolution naturally expanded downwards to unity (i.e., mean period of Earth) and outwards to that of Saturn, followed by the inverse velocity Vi. It is this constant that underlies each particular pheidian spiral subject to exponentiation per degree and continuity per revolution in the present treatment.

For example, a six-whorl spiral for Spira Solaris commences at $0^{\circ}$ and extends to $6 \cdot 360^{\circ}=2160^{\circ}$ with the radius vectors per degree generated from the pheidian constant SD1 $\mathrm{Vi}=w=\phi^{2}=2.61803389875$ :

$$
\begin{align*}
& r=k \cdot w^{\mathrm{n} / 360}\left(w=\text { pheidian growth rate, } k=\text { the starting point, } \mathrm{n}=0^{\circ}, 1,2,3, \ldots, 2160^{\circ}\right)  \tag{2}\\
& r=k \cdot 2.61803389875^{(\mathrm{n} / 360)}\left(k=\text { selected choice, } w=\phi^{2}, \quad \mathrm{n}=0^{\circ}, 1,2,3, \ldots, 2160^{\circ}\right) \tag{2k}
\end{align*}
$$

with $1 / 4$ for the $90^{\circ}$ exponent $\left(90^{\circ} / 360^{\circ}\right), 1 / 2$ for $180^{\circ}$ and $360^{\circ}$ per cycle. Thus back to the equiangular square plus all radius vectors for each intervening degree per revolution until $2160^{\circ}$ is reached. The precise number of whorls selected is a matter of practical convenience; in general the lower the expansion rate ( $w$ ) the greater the number of whorls and $360^{\circ}$ cycles required, and vice versa for the larger rates of growth.

## The Pheidian Planorbidae

Although the name "Pheidias" in the present context is provided by William Schooling's Fig. 389 of Sir Theodore Andrea Cook's The Curves of Life (1914:421), the name Pheidian Planorbidae owes its origins to something entirely different. In fact the name originated from early attempts by the writer to fit equiangular spirals to a variety of shells, ammonites, and in particular, the configuration of the earliest ammonite, Psiloceras Planorbis. Still subject to further refinement, the best fit for the latter is a pheidian growth rate of 1.8995476 per revolution, and therefore (perhaps coincidentally) Phi-series relation (9), the planet-to-planet increase in heliocentric distance ( ${ }^{4 / 3}$ ).

## The Pheidian Planorbidae Thirds

In so much as the present inquiry began with the Peirce planetary framework followed by the sequential inclusion of the Fibonacci, Lucas and Phi-series, the Golden Section (or Three-fold number) not only remains the underlying constant throughout, it also incorporates a dynamic quality provided by the Spiral of Pheidias with a growth rate $w$ of 1.61803398875 per revolution. As does the square of this constant, i.e., Spira Solaris $=\phi^{2}=2.61803398875$, with the latter pair also the Phi-series constants for the planet-synodic-planet and planet-to-planet increases of $S$ and $T$, i.e., relations 7 and 8 respectively. All of which are expressed in terms of the inverse velocity (Vi) that increases by a multiplication factor of $\phi^{1 / 3}=1.1739884997$, thus from the lowest value in the set commencing at Mars the inverse velocities are all exponential thirds. Accordingly: $\phi^{1 / 3}, \phi^{2 / 3}, \phi^{3 / 3}, \phi^{4 / 3} \phi^{5 / 3}, \phi^{6 / 3}$ and $\phi^{7 / 3}$ yield growth factors of $w=$ Vi per revolution for seven sequential equiangular spirals with those of Pheidias and Spira Solaris naturally included.


Fig. 3b. The Pheidian Planorbidae. Exponential Thirds, $w=\phi^{1 / 3}$ through $\phi^{7 / 3}$ (not to scale; $w=\phi^{N / 3}$ )
Does this assign the origins of the equiangular spirals to the complex, interactive motions of the planets per se, and and nothing else? Not necessarily, for there still remains an alternate possibility, which is that the very structure of the Solar System is itself a pheidian reflection of larger, a priori set of conditions and that the "three-fold number" is (or may perhaps be), as Aristotle states, "present in all things whatsoever."
As for the Pheidian Planorbidae, they at least provide a relatively narrow focus applied to planispiral ammonites (as Figure 3b shows) plus two additional benefits. Because the three-fold number in this planetary context embraces periods of revolution ( $T$ ), heliocentric distances $(R)$ and orbital velocity $(V r, V i)$, the growth rate is truly present, i.e., it is not limited to the expansion rate per revolution alone, it also provides periodicity. And, even though the periods may intuitively seem to be too high, the range for Vi from Earth to Saturn can be still checked against growth factors and morphospace contours for 405 ammonites assigned by David Raup (1967: 46-48). ${ }^{42}$


Table 2. Phi-Series data (Earth-Saturn), exponents (x), T, R, Vi (Inverse velocity); Raup (1967:48) w distribution.
Particularly relevant here (as Raup explains in his Summary) is the problem of explaining why ammonoids generally have a $W$ value less than 3.0 but greater than 1.25 , and a $D$ value less than $0.65{ }^{43}$ While $V i=w$ from 3.073533 through 1.173985 lies within the range range assigned by Raup as seen by their inclusion in Figure 4 , with the THIRDS also added to the right vertical axis. But although the range for ammonite growth $w$ given by Raup is in keeping with Vi $=\phi^{1 / 3}$ through $\phi^{7 / 3}$, further intermediate data is necessary to fill the logarithmic gaps between the seven additions.


Fig. 4 Pheidian Planorbidae Thirds 1-7 added to Raup morphospace contours (TEXT-Fig-4.1967:48)

## The Pheidian Planorbidae Sixths

This deficiency is initially met by considering the Lucas-based right triangles of Table 3 and the relation: $\sqrt{ }$ Lucas $=T$ as a possible replacement for the $\phi$-Series to increase the number of intervals between each planet. However, this is best achieved by the $1 / 2 \phi$-Series with geometric means Gm1s and Gm2s filling the logarithmic gaps.

| POSITIONS Lucas |  | $\checkmark$ LUCAS $=T$ | $\phi$-Series $=T$ | $\phi^{\mathrm{x}} \mathrm{x}$ | POSITIONS | ${ }^{\mathrm{x}} \mathrm{x}$ | 1/2 $\phi$-Series | $1 / 2 \phi$-Series | 6th |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Earth Synodic |  |  | 1 EARTH | 0 |  |  | (also, $T=V_{i}{ }^{3}$ ) | (w/Revolution) | \# |
|  | , | 1 |  |  | Gm2 | 0.5 | 1.272019649 | 1.083505882 | 1 |
| MARS | 3 | 1.732050808 | 1.618033989 | 1 | MARS | 1 | 1.618098875 | 1.172984997 | 2 |
|  | 4 | 2 |  |  | Gm1 | 1.5 | 2.058171027 | 1.272019649 | 3 |
| Synodic | 7 | 2.645751311 | 2.618033989 | 2 | Synodic | 2 | 2.618033989 | 1.378240772 | 4 |
|  | 11 | 3.316624790 |  |  | Gm2 | 2.5 | 3.330190677 | 1.493331984 | 5 |
| Pheidias | 18 | 4.242640687 | 4.236067977 | 3 | Pheidias | 3 | 4.236067977 | 1.618033989 | 6 |
|  | 29 | 5.385164807 |  |  | Gm1 | 3.5 | 5.388361704 | 1.753149344 | 7 |
| Synodic | 47 | 6.855654600 | 6.854101966 | 4 | Synodic | 4 | 6.854101966 | 1.899547627 | 8 |
|  | 76 | 8.717797887 |  |  | Gm2 | 4.5 | 8.718552381 | 2.058171703 | 9 |
| JUPITER | 123 | 11.09053651 | 11.09016994 | 5 | JUPITER | 5 | 11.09016994 | 2.230040415 | 10 |
|  | 199 | 14.10673598 |  |  | Gm1 | 5.5 | 14.10691408 | 2.416261907 |  |
| Synodic SD1 | 322 | 17.94435844 | 17.94427191 | 6 | SynodicSD1 | 5 | 17.94427191 | 2.618033989 | 2 |
|  | 521 | 22.82542442 |  |  | Gm2 | 6.5 | 22.82546647 | 2.836655227 | 13 |
| SATURN | 843 | 29.03446228 | 29.03444185 | 7 | SATURN | 7 | 29.03444185 | 3.073532624 | 14 |

Table 3b. The Pheidian Planorbidae SIXTHS 1-14, $1 / 2 \phi$-Series Vi from Earth through Saturn.
The new data increase sequentially by $\phi^{1 / 6}=1.08350588217$ (hence the title Sixths). Represented by color-coded line spiral indicators \#2 to \#15 applied across Raup's morphospace contours, the exponential Sixths have also been added to to the right vertical axis in composite Figure 5a with test format 2 solid spirals shown in Figure 5b:

## Distance (D) of Generating Curve from Axis >



Fig. 5a Pheidian Planorbidae Sixths 1-15 added to Raup morphospace contours (TEXT-Fig-4.1967:48)

PHEIDIAN SIXTHS : PHI-SERIES TEST SPIRALS, MARS - SATURN (Format 2)


Fig. 5b Pheidian Planorbidae Spirals: Exponential Sixths, $w=\phi^{3 / 6}$ through $\phi^{14 / 6}$ ( 1.27201965-3.07353262, not to scale; $w=\phi^{\text {N/6 }}$ )

Despite the overall distribution of the Pheidian planorbidae in Figure 5a the latter might still be dismissed as mere coincidence, while the mean inverse velocity (Vi) is both unexpected and little used in modern astronomy. Yet the grouping is nevertheless fundamentally correct. This becomes more complex, however, when the parameters for the Inferior planets are included.

## Pheidian Planorbidae and Inferior planet relative velocities (Vr)

The $1 / 2 \phi$-Series Planorbidae from \#3 to \#16 are shown in Figure 6 with further details $-T, R$ and $V_{i}$ - provided in the upper section of Table 4. The lower section from IMO2 to Venus employs the reciprocal velocity Vr for the Inferior planets. In so much as the reciprocal spirals are identical in form to those of the Superior planets based on Vi , the corresponding $V r$ spirals for the Inferior planets need not be immediately displayed.


Fig. 6. Phedian Planorbidae(Sixths). Equiangular ammonite/nautiloid test spirals 3-16.

| SATURN | Gm2 | SD1 (Syn) | Gm1 | JUPITER | Gm2 | (Synodic) | Gm1 | PHEIDIAS | Gm2 | (Synodic) | Gm1 | MARS |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 29.0344419 | 22.8254665 | 17.9442719 | 14.10691409 | 11.0901699 | 8.71855238 | 6.85410197 | 5.38836170 | 4.23606798 | 3.33019068 | 2.61803399 | 2.05817103 | 1.61803399 |
| 9.44660279 | 8.04661287 | 6.85410197 | 5.838321602 | 4.97308025 | 4.23606798 | 3.60828119 | 3.07353262 | 2.61803399 | 2.23004042 | 1.89954763 | 1.61803399 | 1.37824077 |
| $V_{1}{ }^{1} 3.07353262$ | 2.83665523 | 2.61803399 | 2.416261907 | 2.23004046 | 2.05817103 | 1.89954763 | 1.75314934 | 1.61803399 | 1.49333198 | 1.37824077 | 1.27201965 | 1.17398499 |
| Vr 0.32535851 | 0.35252786 | 0.38196601 | 0.413862419 | 0.44842237 | 0.48586827 | 0.52644113 | 0.57040206 | 0.61803399 | 0.66964346 | 0.72556263 | 0.78615138 | 0.85179964 |
| ${ }^{1}$ Inverse velocity $\mathrm{Vi}^{\prime}=$ Planorbidae 14 through 2 |  |  |  |  |  |  |  |  |  |  |  |  |
| INFERIOR PLANETS (Half Phi-Series) |  |  |  |  |  |  |  |  |  |  |  |  |
| IMO2 | Gm2 | (Synodic) | Gm1 | IM01 | Gm2 | (Synodic) | Gm1 | MERCURY | Gm2 | (Synodic) | Gm1 | VENUS |


| I | 0.03444185 | 0.04381071 | 0.05572809 | 0.07088723 | 0.09016994 | 0.11469794 | 0.14589803 | 0.18558517 | 0.23606798 | 0.30028311 | 0.38196601 | 0.48586827 | 0.61803399 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | $\begin{array}{llllllllllllll}\text { R } & 0.10585816 & 0.12427589 & 0.14589803 & 0.17128210 & 0.20108262 & 0.23606798 & 0.27714026 & 0.32535851 & 0.38196601 & 0.44842237 & 0.52644113 & 0.61803399 & 0.72556263\end{array}$ $\begin{array}{lllllllllllll}V_{i} & 0.32535851 & 0.35252786 & 0.38196601 & 0.41386242 & 0.44842237 & 0.48586827 & 0.52644113 & 0.57040206 & 0.61803399 & 0.66964346 & 0.72556263 & 0.78615138\end{array} 0.85179964$ $V_{i}{ }^{2} 3.07353262$ 2.83665523 $2.618033992 .41626191 \quad 2.23004042$ 2.05817103 $1.899547631 .753149341 .61803399 \quad 1.49333198 \quad 1.378240771 .272019651 .17398499$

${ }^{2}$ Relative velocity Vr for Planorbidae 14 through 2.
Table 4. $1 / 2 \phi$-Series planetary framework ( $\mathrm{x}=7$ to -7 ) and reciprocal velocities Vi : Vr.
The situation with respect to the Solar System is more complex and also more variable. The inclusion of Pheidias is provided by the Mars-Jupiter geometric mean in Table 5.

| SATURN | Gm2 | SD1 (Syn) | Gm1 | JUPITER | Gm2 | (Synodic) | Gm1 | Pheidias ${ }^{3}$ | Gm2 | (Synodic) | Gm1 | MARS |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 29.5351971 | 24.2050488 | 19.8368199 | 15.3426915 | 11.8667298 | 9.65106718 | 7.84909567 | 6.08944437 | 4.72428089 | 3.84220025 | 3.12481478 | 2.42427747 | 1.88079027 |
| 9.55490959 | 8.36765902 | 7.32793094 | 6.17448919 | 5.20260319 | 4.53297746 | 3.94953908 | 3.33466168 | 2.81551043 | 2.45312680 | 2.13738547 | 1.80463018 | 1.52367934 |
| 3.09110168 | 2.89269062 | 2.70701513 | 2.48485195 | 2.28092157 | 2.12907902 | 1.98734473 | 1.82610560 | 1.67794828 | 1.56624609 | 1.46197998 | 1.34336524 | 1.23437407 |
| 0.32350925 | 0.34569891 | 0.36941057 | 0.40243846 | 0.43841928 | 0.46968665 | 0.50318396 | 0.54761346 | 0.59596593 | 0.63846927 | 0.68400389 | 0.74439919 | 0.81012719 |

## INFERIOR PLANETS (Solar System)

| IM0 ${ }^{4}$ | Gm 2 | (Synodic) | Gm 1 | IM 01 | Gm 2 | (Synodic) | Gm 1 | MERCURY | Gm 2 | (Synodic) | Gm1 | VENUS |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |

$\begin{array}{llllllllllllll}\text { T } & 0.03389857 & 0.04288440 & 0.05425221 & 0.07001434 & 0.090355921 & 0.11430748 & 0.14460813 & 0.18662177 & 0.24084183 & 0.26026754 & 0.32001897 & 0.41249404 & 0.61518332\end{array}$
$\begin{array}{lllllllllllll}R & 0.10474200 & 0.12251791 & 0.14331059 & 0.16987312 & 0.201359014 & 0.23553192 & 0.27550435 & 0.32656894 & 0.38709831 & 0.40764301 & 0.46786133 & 0.55413015\end{array} 0.72332982$
$\begin{array}{lllllllllllll}V_{\text {I }} & 0.32363869 & 0.35002558 & 0.37856385 & 0.41215667 & 0.448730447 & 0.48531631 & 0.52488509 & 0.57146210 & 0.62217225 & 0.63846927 & 0.68400389 & 0.74439919\end{array} 0.85048799$
$\begin{array}{lllllllllllllll}\operatorname{Vr} & 3.08986539 & 2.85693404 & 2.64156235 & 2.42626182 & 2.228509355 & 2.06051184 & 1.90517892 & 1.74989731 & 1.60727194 & 1.56624609 & 1.46197998 & 1.34336525 & 1.17579556\end{array}$
${ }^{3}$ The period (T) for Pheidias is the MARS-JUPITER geometric mean. ${ }^{4}$ Period for IMO 2 from the mean of the Mercury-Venus, IMO 1-Mercury reduction factors.
Table 5. The modern Solar System, (T,S, Gms), ( $R$ ) and reciprocal velocities Vi : Vr.
Lastly,inter-relationships between the Inferior and Superior planets for the unmodified Phi-series are as follows:

| PLANETS N Synodics \# | MODERN $T$ (Julian Years) | x | $\begin{aligned} & \text { i-series } T, S \\ & \text { (Years) } \end{aligned}$ | Phi-series (R) Distance (a.u.) | Phi-series (Vi) Inverse Velocity | Phi-series (Vr) <br> Velocity (Ref.1) |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Uranus 2 | 83.7474068 | 9 | 76.01315562 | 17.94427191 | 4.236067977 | 0.236067977 |
| Synodic 3-2 | 45.3602193 | 8 | 46.97871376 | 13.01969312 | 3.608281187 | 0.277140264 |
| Saturn 3 | 29.4235194 | 7 | 29.03444185 | 9.446602789 | 3.073532624 | 0.325358512 |
| Synodic 4-3 | 19.8588721 | 6 | 17.94427191 | 6.854101966 | 2.618033989 | 0.381966011 |
| Jupiter 4 | 11.8565250 | 5 | 11.09016994 | 4.973080251 | 2.230040414 | 0.448422366 |
| Synodic 5-4 | 7.84767877 | 4 | 6.854101966 | 3.608281187 | 1.899547627 | 0.526441130 |
| Pheidias 5 | 4.72214968 | 3 | 4.236067977 | 2.618033989 | 1.618033989 | 0.618033989 |
| Synodic 6-5 | 3.12552908 | 2 | 2.618033989 | 1.899547627 | 1.378240772 | 0.725562630 |
| Mars 6 | 1.88071105 | 1 | 1.618033989 | 1.378240772 | 1.173984997 | 0.851799642 |
| Earth/Syn 7-6 | 0.91422728 | 0 | 1.000000000 | 1.000000000 | 1.000000000 | 1.000000000 |
| Venus 7 | 0.61518257 | -1 | 0.618033989 | 0.725562630 | 0.851799642 | 1.173984997 |
| Synodic 8-7 | 0.39580075 | -2 | 0.381966011 | 0.526441130 | 0.725562630 | 1.378240772 |
| Mercury 8 | 0.24084445 | -3 | 0.236067978 | 0.381966011 | 0.618033989 | 1.618033989 |
| Synodic 9-8 | 0.14474748 | -4 | 0.145898034 | 0.277140264 | 0.526441130 | 1.899547626 |
| IMO 19 | 0.09041068 | -5 | 0.076806725 | 0.201082619 | 0.448422366 | 2.230040414 |
| Synodic 10-9 | (0.0556507) | -6 | 0.055728090 | 0.145898034 | 0.381966011 | 2.618033989 |
| IMO2 10 | (0.0344447) | -7 | 0.040434219 | 0.105858161 | 0.325358512 | 3.073532624 |

Table 6. Modern Periods ( $T, S$ ) and Phi-series, $x, T, R$, Velocity Vi (Inverse) and Vr (relative), IMO 2 to Uranus.


Fig. 7. Spirasolaris from Mercury to Mars, the three-fold number and unmodified Phi-series.
whereas the actual deficiencies in the Solar System between Mars and Jupiter also need to be addressed.

## Planorbidae 3 through 14 and the Mars-Jupiter Gap

The complete set of Pheidian planorbidae from 3-14 provides a reasonable fit with the Raup morphospace contours but in the Solar System the absence of a body between Mars and Jupiter also excludes the synodic positions and the associated Gms generated by the theoretical $1 / 2 \phi$-Series. Nevertheless, despite such deficiencies adjustments can be calculated for the Solar System, resulting in solid spirals $3 b, 4 b$ and $7 b$ with relatively small positive changes for the final five positions from Jupiter outwards:


Fig. 8. Solar System Planorbidae 3-14. Missing: Nos. 5, 6, 8 \& 9; retained/restored: \#3b, 4b and 7b.
In other words, remaining with the $1 / 2 \phi$-Series format, instead of seven successive periods between Mars and Jupiter only three Solar System periods - Gm1, the Mars-Jupiter Synodic and Gm2 between Mars and Jupiter) now exist.

## Planorbidae spirals and Inferior planet relative velocities (Vr)

However, despite the differences between the modern periods of revolution ( $T$ ) for Mars and Jupiter and the $1 / 2 \phi$ Series, the modern estimate for $G m 2 V_{i}$ is 1.72694432 , whereas the comparable value for the extended $1 / 2 \phi$-Series is $w=\phi^{7 / 6}=1.75314934$. In addition, although Pheidias and associated periods on either side are absent, among extinct ammonites examples exist that approximate the mean relative velocity Vr of Mercury $=1.607281127$ versus the missing inverse velocity $V i=\phi(1.61803398875)$ of the absent Pheidias. Also noteworthy are the similarities for \#7 (1.75314931) versus 1.7498973 , and 1.7269069 in the modern Solar Solar. Relative velocities (Vr) for the inferior planets and the corresponding equiangular spirals (shown as involutes in Fig. 9) are as follows:


Fig. 9. Comparable Solar System Velocities (Vr) for the Inferior Planets [Venus] to IMO2 (\#3i to 14i) shown with relative velocities ( $V r$ ) of interest emphasized in red.

## The Mars-Jupiter Gap revisited

The absence of a suspected body between Mars and Jupiter, associated periods and the theoretical configuration for the $1 / 2 \phi$-Series compared to the present Solar System nonetheless includes the retention and/or "restoration" of planorbidae \#7 (Gm2), albeit with a different origin as shown in the inset table in Figure 10.


Fig. 10. The Mars-Jupiter Gap, planorbidae \#6 and the Solar System; possibly relocated \#7and/or \#7i ?
In short, planorbidae \#5, \#6, \#7, \#8 and \#9 are no longer applicable, with only the Mars-Jupiter Synodic remaining, along with associated real-time intervals between the two planets. Plus, perhaps not entirely justified, inclusion of data from the inferior planets, that of Mercury in particular.

Numerous questions therefore arise concerning not only the full set of planorbidae, but also what can be made of the occurrence of spirals belonging to \#6 (Pheidias) and \#7 in particular among the defunct ammonites and their apparent existence in the Solar System. Furthermore, pentagonal, hexagonal, hexakaidecagonal and icosagonal figures occur ( i.e., 16 and 20 septa per revolution for the latter pair).

## Initial Tests

As far as the initial tests of pheidian spirals applied to ammonites and shells are concerned there remain a number of qualifiers before commencing, e.g., the following supplied by Peter Ward (1992: 85):47

Nautiluses (and apparently the ammonites as well) were not creatures that grew throughout their lives. Like humans, they reached a certain adult size and then quit growing. The slowing of growth immediately preceding the final adult size is marked by changes in the spacing of the last two or three septa formed within the shell and by changes in the shape of the outer shell wall. (italics supplied)
It is at this point, however, that David Raup's treatment of ammonoid spirals comes to mind, particularly his initial range for $w$ from 1.25 to 3 , especially since the inclusion of both Gm 1 s and Gm 2 s to the Pheidian planorbidae Sixths permits the following comparison with the growth factors and the equiangular spirals determined by the latter:


Fig. 11.Test spirals within the Raup range $w=1.25$ to 3.00 and comparable planorbidae. Saturn excluded, the first five planorbidae are all successive odd-numbered geometric means (GM1, GM2: \#3, \#5, \#7, \#9 and \#11).

Questions which arise now are why Raup was unable to proceed further with this promising line of inquiry with its numerical progressions. And also, why the golden ratio/golden section did not surface during the investigation at least as an approximation via the least error geometric mean between 1.5 and 1.75, i.e., 1.6201852 followed in turn by the generally low error GMs for the rest of his estimates given in Table 6 below.

| ASSIGNMENTS | Exp. $x$ | $1 / 2 \phi$-Series $T$ | $1 / 2 \phi$-Series $R$ | 6 Ths | $1 / 2$ \$-Series Vi | Raup w/Gm | Diff (\%) |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| EARTH (Unity) | 0 | 1 | 1 |  | 1 |  |  |
| GM2 | 0.5 | 1.2720196495 | 1.1739849967 | 1 | 1.083505882 | ? |  |
| MARS (Vi) | 1 | 1.6180339887 | 1.3782407725 | 2 | 1.172984997 | ? |  |
| GM1* | 1.5 | 2.0581710273 | 1.6180339887 | 3 | 1.272019649 | 1.25 | -2.20\% |
| Synodic | 2 | 2.6180339887 | 1.8995476270 | 4 | 1.378240772 | 1.3693 | -0.89\% |
| GM2* | 2.5 | 3.3301906768 | 2.2300404146 | 5 | 1.493331984 | 1.5 | 0.67\% |
| \{ PHEIDIAS Vi \} | 3 | 4.2360679775 | 2.6180339887 | 6 | 1.618033989 | 1.62018 | 0.22\% |
| GM1* | 3.5 | 5.3883617041 | 3.0735326237 | 7 | 1.753149344 | 1.75 | -0.31\% |
| Synodic | 4 | 6.8541019662 | 3.6082811871 | 8 | 1.899547627 | 1.9034 | 0.38\% |
| GM2* | 4.5 | 8.7185523808 | 4.2360679775 | 9 | 2.058171703 | 2 | -5.82\% |
| JUPITER (Vi) | 5 | 11.090169944 | 4.9730802506 | 10 | 2.230040415 | 2.2361 | 0.60\% |
| GM1* | 5.5 | 14.106914085 | 5.8383216016 | 11 | 2.416261907 | 2.5 | 8.37\% |
| Synodic | 6 | 17.944271910 | 6.8541019662 | 12 | 2.618033989 | ? |  |
| GM2 | 6.5 | 22.825466466 | 8.0466128743 | 13 | 2.836655227 | ? |  |
| SATURN (Vi) | 7 | 29.034441854 | 9.4466027887 | 14 | 3.073532624 | 3 | -7.35\% |

*Comparable $1 / 2 \phi$-Series $T$ GMs for the Raup growth factors $w=1.25-3$.
Table 6. Pheidian Planorbidae SIXTHS 1-14, ½ $\phi$-Series Vi and the Raup growth factor estimates, $w=1.25$ to 3 .
On the other hand, it is a major step to embrace planetary motion in the first place, not to mention such counterintuitive concepts as inverse velocity in the second, especially without historical guidance. This said, it was only a year after Raup's 1967 paper that Nicole Oresme's Le Livre du ciel et du monde was published in Madison in 1968. It was this work concerned with both the commentary by Averroes on Aristotle's "On the Heavens" and additional insights by Nicole Oresme which supplied the weighty and erudite statement given earlier. It is worth repeating:

Said Aristotle, never-failing friend of Truth: The three-fold number is present in all things whatsoever, nor did we ourselves discover this number, but rather, nature discovers it for us.

## Test Procedures adopted for Ammonites and Nautiloids

There is one further point concerning Raup's pioneering analyses, which is that his methodology for establishing the growth factor $w$ must have been relatively limited with some degree of mechanical measurements required to arrive at his given estimates for growth per revolution. The present treatment which follows - which is likely already superceded by more sophisticated methods - nonetheless produces accurate, double-precision equiangular radius vectors per degree for successive whorls, thus 2160 successive data points for 6 -whorl spirals, and 2880 data points for 8 -whorl spirals, etc. Furthermore, based on the same data, the various figures included also maintain the same accuracy, being the specific points which correspond to $120^{\circ}$ for the triangle, $90^{\circ}$ for the square, $72^{\circ}$ the pentagon, $60^{\circ}$ for the hexagon and $45^{\circ}$ for the octagon, etc., and so on down to the $1^{\circ}$ points of the spirals themselves. Thus the following colour-coded equiangular test spirals for figures that occur among ammonites:


Fig. 12. Growth factors ( $w$ ) with $3,4,5,6,8,12,16$ and $20 X$ configurations for the Pheidian Sixths and Solar System.
Additional information is also provided by various integral figures known to occur among ammonites that extend to the outermost regions shown in Figures 12 and 13, albeit to a far lesser extent for the Solar System equiangular spirals of potential interest in the latter group.


Fig. 13. Solar System growth factors ( w ) and equiangular spirals of additional interest.

In Figure 13, MSM is the mean synodic month in days with the relative velocity Vr finally expressed with respect to unity; see Tables AP1 and AP 2 in Part One for further details. Included earlier in Figure10, the Mars-Jupiter Gm2 of 1.72697078 associated with the Mars-Jupiter Gap is at this stage perhaps best examined with respect to the spiral configurations of certain sea-shells. One shell in a particular in this context is Thatcher Mirabilis with a provisional growth factor of $w=\phi^{7 / 6}=1.75314934$, thus an equiangular spiral in close proximity to $w=1.72697078$ and therefore a potential replacement for the region of special interest.

## Initial test procedures for ammonites and nautiloids

The superb line drawings of ammonites in the Treatise on invertebrate paleontology ${ }^{48}$ provide a major resource for the refinement of test techniques once the last septa is repositioned to the lower, free-swimming position. At this point, however, it is necessary to acknowledge that this is a highly specialized field, even before such topics as time scales, variants and extinctions add further complexity. This said, the limited range of the growth factors ( $w$ ) along with the similarities in form and standard orientation suggests that planispiral ammonites nevertheless provide a workable test set for the spirals introduced in Figures 5b and 6 for Pheidian planorbidae numbers 3 through 14.

Lastly, in fitting all double-precision generated planorbidae spirals to test subjects, apart from colors, density and line widths, modifications were limited to scaling and rotation alone. As for the fit, the proportion and the form of test subjects were equally inviolable, each confined to reorientation and scaling. For spatial considerations the test subjects were also displayed in the same size. The best fit was taken to be the middle spiral of three consecutive planorbidae as shown in Figure 14 and the larger, more complex drawing of Manticoceras assigned to Figure 15. For the quadruple test set planorbidae \#9 is too small (-), \#10 the best fit and \#11 too large (+).


Fig. 14. Manticoceras ${ }^{50}$ and planorbidae \#9 (-), \#10 (Jupiter Vi, $w=2.230004041$ ) and \#11 (+).


Fig. 15. Manticoceras ${ }^{49}$ and planorbidae \#10 (Jupiter Vi, w= 2.230004041).

## Initial Tests

Sequential Pheidian Planorbidae applied to ammonite drawings in the Treatise on Invertebrate Paleontology (1957) are shown in Figure 16A_D. The inverted, scaled drawings for this initial test feature growth factors (w) from \#6 (the spiral of Pheidias) through Spiral Solaris (\#12) plus \#13. The first (15A) is a lone example which also shows a vestigial "square" figure in its structure, as does Figure 16 which follows (albeit for planorbidae \#7), while integral pentagonal, hexagonal and octagonal figures also follow in due order.


Fig. 16A-D. Sequential Pheidian Planorbidae applied to technical ammonite drawings in the Treatise on Invertebrate Paleontology (1957) with best fit growth factors (w) for planorbidae \#6 through \#13. (all reoriented and rescaled).

Next, the process is applied to high quality photographs of ammonites which feature integral squares, pentagons hexagons and octagons (Figures 16 through 19). Once again, the preferred spirals and figures are middle values
between three consecutive Planorbidae with the first spiral marginally too small and third marginally too large.

$\phi_{k}=6 / 6, w=1.618033$
$\phi_{k}=7 / 6, w=1.75315$
$\phi_{k}=8 / 6, w=1.89954$
Fig. 17. Calliphylloceras biciolae (Meneghini 1874) juv. ${ }^{55}$ Pheidian spirals/squares: \#6 ( - ) <best fit \#7>; \#8 (+).


Fig. 18. Phylloceras (Hypophylloceras) paquieri Sayn. ${ }^{56}$ Pheidian spirals/pentagons: \#9( - ) <best fit \#10>; \#11(+).

$\phi_{k}=7 / 6, w=1.7531$
$\phi_{k}=9 / 6, \quad w=2.0582$
Fig. 19. Puzosia aff. quenstedti (Parona \& Bonarelli 1897). ${ }^{57}$ Pheidian spirals/hexagons: \#7(-) <best fit \#8>; \#9(+).


Fig. 20. Sallfelldiella (Salfelldiella) guettardi RASPaill 1831).55 Pheidian spirals/Octogons: \#7 ( - ) <best fit \#8>; \#9 (+).

Part of a larger set, the preceding assignments served well enough to expand testing to include more high quality plan and side-view photographs of ammonites for the pheidian spirals shown next in Figures 21 through 22:


Fig. 21. Metaplacenticeras subtilistriatum (Jimbo 1894) ${ }^{59}$ with planorbidae \#11 ( - ) <best fit \#12> and \#13 (+).


Fig. 22. Desmoceras latidorsatum (Michelin 1938) ${ }^{60}$ with planorbidae \#8 ( - ) <best fit \#9> and \#10 (+).



Plus, it would seem, even when there are larger overlaps for the thicker ammonites, the overlaps nevertheless appear to maintain the same pheidan form as the parent spiral, e.g., Figures 23 and 24:


Fig. 24. Hauericeras gardeni ${ }^{62}$ with planorbidae \#7( - ), <best fit \#8> and \#9(+).
Where ammonites are damaged or showing pronounced changes at the outermost septa it is possible to find a provisional best-of-three fit for certain examples from the inner-displayed spiral alone, e.g., for Stephanoceras:


Fig. 25. Stephanoceras. sp. ${ }^{63}$ Inner spirals: planorbidae \#7 ( - ) < best fit \#8> with \#9 slightly too large (+).
And finally, best-fit assignments that include Spirasolars (\#12), the missing Pheidias (\#6), and missing \#7:

$\phi_{k}=12 / 6, w=2.618034$
$\phi_{k}=6 / 6, w=1.618034$
Fig. 26A-C. A: Outline, Spirasolaris,
B: Whorl-to-whorl spiral, Pheidias,\#6
$\phi_{k}=7 / 6, w=1.75315$
C: Whorl-to-whorl spiral, \#7.
A. Gaudryceras denmanense (Whiteaves 1901). ${ }^{64}$
B. Septimaniceras zittel (Oppel 1862)(M). ${ }^{65}$
C.. Nannolytoceras pygmaeum (d'Orbigny 1845 . ${ }^{66}$

Photographs: Hervé Chatelier, Ammonites.fra (\#0702; 0702v reduced). Photographs: Hervé Chatelier, Ammonites.fra (\#0147; 0147v reduced). Photographs: Hervé Chatelier, Ammonites.fra (\#0603; 0603v reduced).

At which point, using the above methodology, back to the growth factors ( $w$ ) for four adjacent positions: 8/6 9/6 and Solar System substitutes for 10/6 and 12/6, i.e., for Jupiter and Jupiter-Saturn SD1 Vi = Spirasolaris.


Fig. 27A-D. Ammonites, best fit Sixths ( A \& B ); Solar System Vi for Jupiter ( C ) and Jupiter-Saturn Vi ( D ) SD1 = Spirasolaris.
A. Macrocephalities verus Buckman 1922. ${ }^{67}$
B. Paracladiscites ${ }^{68}$
C. Beudanticeras laevigatum (Sowerby 1827). ${ }^{69}$
D. Metaplacenticeras subtilistriatum (Jimbo 1894). ${ }^{70}$

Photograph: Hervé Chatelier, Ammonites.fra (\#0026).
Photograph: Hervé Chatelier, Ammonites.fra (\#0766). Photograph: Hervé Chatelier, Ammonites.fra (\#0766).
Photograph: Hervé Chatelier, Ammonites.fra (\#0258).

## "Triangular" Ammonites, Radiolarians and Diodom(s)

Despite their rarity, a small sample of triangular ammonites are nonetheless included in the Treatise on invertebrate paleontology, e.g., line drawings of Soliclymenia, which, with a lightened grey-scale (but otherwise unchanged) can checked against transparent, scalable and rotatable overlays of the three closest equiangular spirals and associated equi-triangles.


Soliclymenia, $\phi^{1 / 2}$ equi-triangles.
Soliclymenia, $\phi^{2 / 3}$ equi-triangles.
Soliclymenia, $\phi^{5 / 6}$ equi-triangles.
Fig. 28. Triangular ammonite Soliclymenia; ${ }^{71}$ limited fit equiangular triangles for $\phi_{k}=2 / 3$ (4/6), $w=1.37824077$ between $\phi_{k}=1 / 2(3 / 6), w=1.272019649$ and $\phi_{k}=5 / 6, w=1.49333989$ (parent equiangular spirals omitted).

## Radiolarians and Diodoms

## Radiolarians

It is at this point, however, in view of the scarcity of triangular ammonites and similarities in shape elsewhere that the inquiry now expands to include microscopic organisms and added complexity associated with natural growth. Fortunately, however, the configuration of one example in particular lends itself readily to the task (as shown next in Figure 29) by the fit for a micro-photograph of Late Triassic radioarian Sarla. Here the exponential growth is more readily apparent from the angle and scale of the left side extremity, with the smaller side on the right and largest vertical component together providing the orientation for a plan-view of anti-clockwise growth.

All that remains is the determination of a centre from the intersection point of the three lines (B) dropped from the extremities with the resulting spiral and triangular figures again the growth rate of $\phi^{2 / 3}, w=1.37824077$.


Fig. 29. Radiolarian Sarla. (Late Triassic). ${ }^{72}$ Equiangular Planorbidae spiral \& triangles $\phi_{\mathrm{k}}=2 / 3, w=1.37824077$.


Fig. 30. Radiolarian Chariottea amurensis. ${ }^{73}$ Equiangular Planorbidae spiral \& triangles $\phi_{k}=2 / 3, w=1.37824077$.


Fig. 31 A-C. Radiolarians ${ }^{74,75}$ and Equiangular triangle/square/spirals, $\phi_{k}=2 / 3, w=1.37824077$.
Thus, despite wide variations in size and shape, certain microscopic radiolarians and diodoms can be assigned equiangular spirals replete with integral equiangular figures. Accordingly, the four micro-photographs shown here were re-oriented to synchronise with anticlockwise motion and growth ( $w$ ) per revolution for best-fit spirals and equiangular figures in the same manner used for ammonites. Therefore they also require a center, provisionally the junction point of lines extended inwards from each extremity towards their common meeting point as applied in Figures 29 and 30. The "square" configuration in Figure 31c was simpler to test since the centre is provided by the horizontal and vertical axes of the test spirals whi9ch were included for such purposes.
The sudden shift to invoke equiangular spirals with such a small growth factor requires explanation, whatever its its origins might be. Clearly, any equiangular spiral would now have a much smaller growth rate than the Pheidian Thirds and the Sixths. Even so, for continuity Pheidian growth was retained with test spirals (plus triangular figures)
initially involving the Twelfths, namely $\phi^{1 / 24}(w=1.034969827), \phi^{1 / 12}(w=1.040915886), \phi^{1 / 16}(w=1.030532583)$ and $\phi^{1 / 15}$ ( $w=1.032600924$ ) then finally rounded out by $\phi^{1 / 9}(w=1.054923213)$. Plus, from Plato's Timaeus and for good measure, the growth factor which results from the ratio 256/243 ( $w=1.053497942$ ).

## Diodom(s)

The third example concerns the semi-related shape of a Diodom which already has a distinct center, but no obvious suggestion that a spiral of any kind is necessarily involved in its structure. Apart, that is, from the slight difference in lengths for the indicated $120^{\circ}$ divisions, which prove to be sufficient to at least reorient the diodom as done for the radiolarians. Plus, by way of the geometric mean, an equiangular spiral based on $\phi_{k}=1 / \sqrt{ } 72(0.117851130197758)$ with a growth factor $(w)$ of 1.05835028137695 .


Original orientation \& center.


Test configuration

$\phi_{\mathrm{k}}=1 /(\sqrt{ } 72)$ Spiral, $w=1.0583502814$


Diodem $/ w \phi_{\mathrm{k}}=1 /(\sqrt{ } 72)$ Spiral.

Fig. 32. Diodom Symbolophorus amblyoceros ${ }^{76}$ and equiangular spiral/triangles, $\phi_{\mathrm{k}}=1 /(\sqrt{ } 72), w=1.0583502814$.
In wider natural contexts, however, two diverse configurations immediately come to mind, namely individual snowflakes and the hexagonal cells grouped together in honeycombs. Both have pluses and minuses in terms of investigation, and so nearly perfect are their structures that neither suggest a spiral is necessarily present at all. Or perhaps more to the point, such spirals would be nearly circular with extremely small values for the growth factor per revolution. This is where the discussion returns to microscopic organisms, beginning with radiolarians, where it was perhaps fortuitous that the Sarla example suggests both logarithmic expansion and a possible spiral form among a bewilderingly wide range of other configurations. To a lesser extent the same may be said of the second and third examples, with the equiangular "squares" usable, although somewhat irregular.

## Bi-Polar Figures

Clearly, this expansion has moved far beyond the initial investigation into the fit of sequential pheidian parameters to planispiral ammonites. Moreover, in spite of the suitability of generally complete and symmetrical radiolarians examined here, many examples in GSC 496 and the Treatise on invertebrate paleontology are unusable for this task due to damaged extremities or relatively unsymmetrical forms. However, there are still "square" and "hexagonal" radiolarians plus other "triadic" examples including relatively robust "bi-polar" forms,e.g., the radiolarian Chariottea harbridgensis ${ }^{74}$ similar to that discussed next.


Fig. 33. Radiolarian Chariottea harbridgensis ${ }^{77}$ Equiangular Planorbidae spiral (B) $\phi_{k}=5 / 6, w=1.4933898$ and main spiral (C and D) from the Pheidian Twelfths - $\phi_{k}=5 / 12, w=1.2220196$.

The micro-photographs of radiolarians tested so far are from GEOLOGICAL SURVEY OF CANADA BULLETIN 496 (E. S. Carter, P. A. Whalen, and J. Guex, 1998) which also includes a number of bipolar figures. But even if the bipolar formats also employ pheidian growth with resulting equiangular spirals, there would appear to be no immediate way to determine the center of the spiral or the growth factor ( $w$ ). Nevertheless, the provisional selection of three equiangular spirals: 1. Primary, $\phi_{k}=1 / 3, w=1.173984997$ (error: $5.26 \%$ ), 2 . Width $B-X, X-B^{\prime}$ (common center), $\phi_{k}$ $5 / 12, w=1.222019633$ (error: $-0.537 \%$, and 3 . Internal at $45^{\circ}, \phi_{k}=9 / 12, w=1.434632715$ (error: $0.767 \%$ ) obtained from measurements of a bi-polar figure included in Fig. 34A by Pessagno, E.A. Jr., and Blome, C.D. 1980:


Fig. 34. Added best-fit equiangular pheidian spirals based on three sets of accurate, intersecting measurements and markers for Upper Triassic Pantanelliinae determined by Pessagno, E. A. Jr., and Blome, C.D. 1980. ${ }^{78}$ The larger spiral has been flipped $180^{\circ}$ about the vertical axis to improve the fit.

The luxury of having a clearly defined, accurate center for this bi-polar figure also provides an opportunity to apply the pheidian planorbidae best of three fit for adjacent or close known examples, which in the present case involves the lower Sixths: $\phi_{k}=1 / 6, \phi_{k}=2 / 6(1 / 3)$ and $\phi_{k}=3 / 6(1 / 2)$. Scaled to fit in each instance from S' to S only one of the latter will also match or come close to the center at $X$, as indeed is the case for $\phi_{k}=2 / 6(1 / 3)$, as seen in Figure 35 where the differences on either side are both relatively small:


Fig. 35. Best fit test for the Pheidian Sixths, $\phi_{k}=1 / 6,2 / 6$ (best fit) and $\phi_{k}=3 / 6$ based on the measurements for Upper Triassic Pantanelliinae determined by Pessagno, E. A. Jr., and Blome, C.D. The spirals are again flipped $180^{\circ}$ about the vertical axis.

At which, having already moved beyond original intentions there remains the matter of hexagons and octagons among ammonites and the most likely Pheidian Planorbidae of interest in this context, ( $\phi^{4 / 3}, w=1.8995476$ ).

## The Pheidian Sixths as Periods ( $T$ ) in Julian years and days

Even though equiangular squares, pentagons, hexagons and octagons are discernable in Figures 17 to 20, they are not clearly defined over a full revolution. Furthermore, recalling the complications raised by Peter Ward concerning ammonite growth and the outermost septa, what correlation there is invariably deteriorates as the last quadrant is reached. Then again, this region is more likely to suffer damage over time in addition, whereas the actual age of the examples may also be crucial, i.e., pertaining to juvenile stages of development during which rapid changes may or not have taken taken place. Exactly how rapid and precisely what changes may have occurred may be debatable, with the suggestion that progression between figures - the triangle, the square, the hexagon and beyond - purely speculative. Nevertheless - as shown earlier - there does indeed exist an ammonite which is triangular in form. ${ }^{67}$

Either way, it now becomes necessary to include time and distance with velocity in terms of pheidian growth in the present astronomical context. In short, to produce the same planorbidae and same separation for Periods ( $T$ ) and $(S)$ from the generation of the Pheidian Sixths from Earth outwards for 1 to 3.073532624 years and also their corresponding radius vectors for the distances $(R)$ and the velocities (Vi) with respect to unity.

Before more detailed examination, in terms of the possible pheidian nature of the Solar System it is necessary to recall that the four major relationships which pertain to increases in the mean sidereal periods of revolution ( $T$ ) and mean heliocentric distances ( $R$ ) - in years and a.u. respectively - are also members of the planorbidae Sixths, i.e., numbers 4, 6, 8 and 12:
\#4: Phi-series Planet-Synodic distance (R) increase: $\quad k=2 / 3$ (1.37824077249), Outwards. Relation (11)
\#6: Phi-series Planet-Synodic periods (S) increase: $\quad k={ }^{1}$ (1.61803398875), Outwards. Relation (8)
\#8: Phi-series Mean heliocentric distance ( $R$ ) increase: $k={ }^{4 / 3}$ (1.89954762695), Planets, Outwards. Relation (9)
\#12: Phi-series Mean periods of revolution ( $T$ ) increase: $k={ }^{2}$ (2.61803398875), Planets, Outwards. Relation (7)
All four are of continuing interest, especially ${ }^{1}$ and ${ }^{2}$ underlying the spiral of Pheidias and Spirasolaris. Moreover, relation (9) also concerns mean distance $(R)$ with $w=1.89954762695$ again associated with both the hexagon and the octagon in Figures 19 and 20. Thus, with the Pheidian Thirds as Phi-series periods derived from successive values from 1 through 7 a new test set including the latter range in years and days is as shown in the following table:

| Relation/resonance | Period $T$ (Years) |  | Period $T$ (Days) | Distance $R$ | Velocity (Vi) | Velocity $(V r)$ |
| :--- | :---: | :--- | :--- | :---: | :---: | :---: |
| EARTH: Synodic | THIRDS | $\#$ | 365.25 | 1 | 1 | 1 |
| Relation $12(V i)$ | 1.172984997 | 1 | 428.79802004 | 1.1128629857 | 1.054923213 | 0.947936293 |
| Relation $11(R, S)$ | 1.378240772 | 2 | 464.60517698 | 1.2384640249 | 1.112862986 | 0.898583215 |
| Relation $8(T, S)$ | 1.618033989 | 3 | 503.40244215 | 1.3782407725 | 1.173984997 | 0.851799642 |
| Rel. (9) FOURTH $4: 3$ | 1.899547627 | 4 | 693.80977074 | 1.5337931411 | 1.238464025 | 0.807451795 |
| $\quad$ MAJOR SIX 5:3 | 2.230040415 | 5 | 814.52226142 | 1.7069016144 | 1.306484449 | 0.765412862 |
| Rel. (7) OCTAVE 2:1 | 2.618033989 | 6 | 956.23691439 | 1.8995476269 | 1.378240773 | 0.725562630 |
|  | 3.073532624 | 7 | 1122.6077907 | 2.1139362436 | 1.453938184 | 0.687787150 |

Table. 7. The Pheidian planorbidae $T$, Thirds, \#1 to \#7 in Julian years and days


Fig. 3y. The Pheidian Planorbidae in Years. Thirds, $w=\phi^{1 / 3}$ through $\phi^{7 / 3}$ (not to scale; $w=\phi^{N / 3}$ )
This provides a useful line of inquiry, e.g., using relation (9) with $T={ }^{4 / 3}=1.89954762695$ and the Julian year of 365.25 days the correspondingly period is 693.8097707 days per revolution with the mean periods for each of the six or eight partitions obtained by simple division:

Hexagon (6/side): 115.634961791 days per septum
Octagon (8/side): 86.726221343 days per septum

These results are recognisable as periods associated with the combined motions of both Earth and Mercury, i.e., the Julian year, synodic relation (1) and the modern period ( $T$ ) for Mercury produce the accurate synodic period (S)

$$
\begin{array}{lr}
\text { Mercury-Earth synodic }(S): & 115.876693996 \text { days } \\
\text { Synodic relation }(1): & 365.249999988 \text { days } \\
\text { Mercury }(\text { mean period } T): & 87.968435362 \text { days }  \tag{6}\\
\text { with } 693.80977074 / 365.25=1.889547627 \ldots\left({ }^{4 / 3}\right)
\end{array}
$$

The value for Mercury in relation (4) is on the low side, but that said, it is also closer to the comparable phi-series period for this position ( 86.726221343 days versus 86.22382878 days obtained from the product ${ }^{-3} \mathrm{JYR}$ ). Whether these two periods per section/septum are valid remains an unresolved issue at this stage of the investigation.

Whereas, despite their minor differences, the identical procedures for relations (3) and (4) produces a significant result recognisable as a close approximation for the luni-solar eclipse cycle: ECY of 346.620107 days (modern) and 346.619576 days (Babylonian) obtained from the mean synodic (MSM) and mean draconic month (MDRA):

$$
\begin{array}{ll}
\text { Hexagon (per side): } \quad 115.634961791 \text { days per septum } \\
\text { Synodic relation (1): } & 346.904885372 \text { days per cycle } \\
\text { Octagon (per side): } \quad 86.726221343 \text { days per septum } \tag{4}
\end{array}
$$

As for associated planorbidae \#8 for the two figures ( $\phi_{k}=(8 / 6), 4 / 3, w=1.8995476269$ ), a half of 693.80977074 days per revolution is also 346.9048854 days. More immediately, however, despite relatively small masses, the positions of Earth, Venus, Mercury, and below the latter theoretical IMO 1 (of unknown mass and uncertain presence) can all be included. Thus for the planorbidae of primary interest - \#2 and \#4 from the Thirds (\#6 and \#8 from the Sixths) - in the present context, remaining with the product of of the latter and the Julian year, the following table of Fibonacci divisors applied to the 693-day period with modern estimates for comparison is:

| Divisor Periods (days) | Modern (T,S) | Assignments (T,S), Cycles |
| :---: | :---: | :---: | :---: |
| Base | 693.8097707 | Julian Yr 365.25 = Earth, Revolution $T$ (days) |
| $\mathbf{2}$ | 346.9048854 | $346.604554 \sim$ Eclipse Cycle EYC (days) |
| 3 | 231.2699236 | $224.695434 \sim$ Venus, Sidereal $T$ (days) |
| 5 | 138.7619541 | $144.566223 \sim$ Mercury-Venus Synodic S |
| (6) | 115.6349618 | $115.876694 \sim$ Mercury-Earth Synodic S |
| $\mathbf{8}$ | 86.72622134 | $87.9684354 \sim$ Mercury, Sidereal $T$ (days) |
| 13 | 53.36998236 | $52.8177728 \sim$ IMO1-Mercury Synodic $S$ |
| (18) | 38.54498726 | $38.6843214 \sim$ IMO1-Venus Synodic $S$ |
| -- | 33.99877856 | $33.9988938 \sim$ MSM-Venus Synodic Si |
| $\mathbf{2 1}$ | 33.03856051 | $33.0025 \quad$ = IMO1, Revolution $T$ (days) |
| -- | 29.53059414 | 29.5305895 = MSM, Mean synodic month |

Table 26. Triangular, pentagonal, hexagonal, octagonal divisors and the Inferior planets plus IMO1.Base period of 693.809770744 days from \#8: $\phi_{k}=8 / 6$ or (4/3), $w=1.89954762698 \cdot J Y R(365.25$ days $)$.

Technically, the inclusion of the mean synodic month is positionally incorrect, yet despite the disparity and further complexity which arises from its concurrent motions about Earth, in terms of order the period is numerically correct. In other words, the numerical positions from Earth inwards according to the periods are Earth-Venus-Mercury-Moon.
Thus Earth, Venus and Mercury are joined by IMO1 plus, not unsurprisingly, the motion of the moon, albeit initially the approximate 29.5-day mean synodic month (MSM; Babylonian 29;31,50,78,20 days) and the shortest associated month, the draconic (MDRA). Furthermore, the 411-day anomalistic cycle involving MSM and anomalistic month (MAN) can be similarly investigated by reversing the period process.

The fact is, however, that almost all the present attempts to expand on the occurrences of spirals - pheidian or otherwise - are hampered by orientation, and above all, periodicity. Plus another area of complexity arising from three-dimensional form and growth. On the other hand, at least, certain growth factors seem to occur more often than others, e. g., $\phi_{k}=(2 / 3), w=1.37824077$ among radiolarians, and $\phi_{k}=(4 / 3), w=1.899547627$ discussed above and earlier with respect to hexagonal and octagonal features among ammonites.

## Pheidian Thirds and Spirals among particle tracks in bubble chambers

In view of the relatively small numbers of examples tested to date, additional searches for spiral growth based on $\phi_{k}=(4 / 3), w=1.899547627$ were undertaken. The searches more recently included recognizable examples of spirals encountered among particle tracks in bubble chambers, with those investigated by Syed Waqar Ahmed ${ }^{79}$ of the

National University of Computer and Emerging Sciences, Karachi, Pakistan of interest. Augmented by Wolfram ${ }^{80}$, this research was made freely available on the Internet, which in the present instance included a description of the experiment and the methodology applied to the fitting of spirals in this context. The results as given in the text are shown below in Figure 36.


Fig. 36. The fitting of spirals to those in bubble tracks described and demonstrated by Syed Waqar Ahmed.
Next, for comparison with the standard colours used for the Pheidian Planorbidae, the blue background of the original figure has been replaced by black with inner titles $A, B, C$, the orange spiral, and three-spiral inset in Figure 37AC added to aid the following commentary.


Fig. 37AC. Pheidian Planorbidae and spirals in bubble tracks. Thirds, $w=\phi^{4 / 3}, \phi^{5 / 3}$ and $\phi^{7 / 3}$ ( not to scale; $w=\phi^{N / 3}$ ) (The larger spiral was provisionally assigned a growth rate of $w=\phi^{6}\{17.94427191: 1\}$ per revolution but not included in the inset as such).

## Remarks

First of all, the blue spiral (A) is clearly recognizable as the spiral of current interest - $\phi^{4 / 3}$ with $w=1.8995476$.
Secondly, the blue-green spiral (B) is found to be the next Third, i.e., following $\phi^{4 / 3}$ is $\phi^{5 / 3}$ with $w=2.2300404$.
Thirdly, the green spiral(C) also belongs to the Thirds, but no longer sequential, i.e., $\phi^{7 / 3}$ with $w=3.0735326$.
Fourthly, the orange spiral (top left addition) is given an uncertain assignment of $\quad \phi^{6}$ with $w=17.944272$.
Although not a problem per se, there is nevertheless something puzzling about the inclusion of equiangular spirals without identification in modern works, especially those of scientific or mathematical nature. Yet as shown next, there are three examples of the spiral under consideration ( $\phi^{4 / 3}, w=1.8995476$ ) in an "artistic" format published in 2001 by mathematician Ian Stewart.


Fig. 38. The Pheidian spiral $\phi^{4 / 3}, w=1.8995476$ and spirals with geometric patterns in What Shape is a Snow Flake? Magical Numbers in Nature included by lan Stewart. ${ }^{81}$

Nor do such additions stop here either; with the self-same spiral again in a similar artistic representation ${ }^{82}$ which might suggest - intentionally or otherwise - that the ratio between the two outermost whorls of these major spirals may be of interest in addition to the esoteric beauty of the images, and from such a start it is just possible that the Golden Ratio itself might ultimately emerge in a setting already geared towards scientific expansion.

A fanciful notion perhaps, but given how close David Raup got to the Phi-series planorbidae shown earlier in the Figure 11 comparisons ("Test spirals within the Raup range $w=1.25$ to 3.00 . . "), and the 22 years since Stewart's inclusion of this fundamental spiral, we might well benefit from an occasional hint or prod now and again.

Then again, in the last month of the year 2023 matters are becoming increasingly disturbing on a global scale although this is not the time for anger or despair against the self-serving excesses enacted by nations, religions, institutions, networks or members at large. Better to follow the cautions offered by Confucius about murmuring against God or raging at Mankind, and continue with the business at hand. Which in the present context was the intimations of Benjamin Pierce's Fibonacci, Lucas and Phi-based planetary framework. Followed n turn by natural expansions generated by the apparent presence of the Fibonacci series and Phi itself in Babylonian astronomical cuneiform texts of the Seleucid Era. Which, although not originally intended, precipitated the need for an optional Excursus on the matter included at the end of this final section, our increasingly troubled times notwithstanding.

As for the past on a wider scale, the extensive range and influence of Babylonian mathematics are discussed in detail by Jorän Friberg in works published in 2005, 2006 and 2007. ${ }^{83-85}$ More recently, inroads into complexities inherent in Babylonian astronomical texts of the Seleucid Era have been made by Mathieu Ossendrijver (2016) ${ }^{86}$ concerning the velocity of Jupiter and an unexplained trapezoid in Babylonian texts for this planet. Furthermore, additional complexities arise from the analysis by Daniel F.Mansfield and N.J.Wildberger (2017) ${ }^{87}$ of Plimpton 322 (a complex Babylonian mathematical text) as an exercise in "Babylonian exact sexagesimal trigonometry."

On a more mundane level it should be explained that the name planorbidae did not stem from the astronomical nature of the two Phi-series under discussion, but earlier attempts to fit planorbidae spirals to shells, ammonites and in particular the configuration of the earliest ammonite, Psiloceras Planorbis. Subject to further refinement, the best fit for the latter is a pheidian growth rate of 1.8995476279 per revolution, and thus (perhaps coincidentally), Phi-series relation (9), the planet-to-planet increase in heliocentric distance ( ${ }^{4 / 3}$ ). This remains to be confirmed or redefined, as do further attempts to fit scalable planorbidae to various organisms and mechanisms including radiolara, diodoms, foramina, shells and ammonites. This said, there are now sophisticated modern studies such as the "Ammon" Database by Paul L. Smith (1986) ${ }^{88}$ and later work by Liang and Smith (1997). ${ }^{89}$

On a far wider level, who knows how extensive and how complex "pheidian order" might ultimately be, the very structures of tornadoes, hurricanes and spiral galaxies included at one extreme and micro particles at the other. Which are, after all, natural and logical expansions of Aristotle's "all things whatsoever," are they not?

Another purpose for including ammonites in the present discussion is the apparent retention of the underlying planetary structures despite omissions and departures from the theoretical model both locally and further afield. This, coupled with the reappearance of "Lazarus" ammonites ${ }^{90}$ eons after extinction suggests that some form of ordering mechanism may be involved. Although not, of course, with respect to this occurrence, the possibility of relevant knowledge in the past comes to fore, augmented by both the Fibonacci and Lucas series in the present study. It remains, however, unclear how much relevance and verity unduly obscure statements in the Chaldean Oracles concerning the "monad," the "dyad," and "Intellectual sections to govern all things, and to order all things not ordered." Or, in the same place, "Fountain of fountains, and of all fountains, the matrix containing all things. ${ }^{91}$
Certainly Plato and Proclus were held in high regard centuries after their passing, with the former's Timaeus also given special prominence despite its scale and complexity. In this respect the opinions of later commentators are also illuminating. Macrobius, for example, in his Commentary on the Dream of Scipio, states that "numbers preceded the World-Soul, being interwoven in it, according to the majestic account in the Timaeus, which understood and expounded Nature herself." ${ }^{92}$ Whereas Vitruvius (ca. $75-25$ BCE) sort fit to include the number 216 (i.e., $6^{3}$ from the senary interval) in the Ten Books on Architecture in an obscure reference to Pythagoras and rules "founded on the analogy of nature." ${ }^{93}$ This is a recurring theme; according to Theon of Smyrna, the Tetractys referred to earlier was "not only principally honored by the Pythagoreans, because all symphonies are found to exist within it, but also because it appears to contain the nature of all things." ${ }^{94}$ Thus all-encompassing assessments embraced by scholars that seem to have continued up to and including the Reformation, but thereafter less publicly.
As for earlier times, consider the insights provided by Archytas ${ }^{95}$ who was reputedly: "The first who methodically applied the principles of mathematics to mechanics: who imparted an organic motion to a geometric figure." Italics are supplied here for one primary reason, which is the linking of mathematics, motion, and ultimately, "Nature," in an organic sense. What follows from this awareness, or if preferred, this theoretical premise, remains conjectural whereas Plato provides both encouragement and a positive conclusion towards the end of the Epinomis, ${ }^{96}$ thus:

> To the man who pursues his studies in the proper way, all geometric constructions, all systems of numbers, all duly constituted melodic progressions, the single ordered scheme of all celestial revolutions, should disclose themselves, and disclose themselves they will, if, as I say, a man pursues his studies aright with his mind's eye fixed on their single end. As such a man reflects, he will receive the revelation of a single bond of natural interconnection between all these problems.
> (Plato, Epinomis 991-992).

Whatever the single bond and the natural interconnection might be, from the contents of many ancient works some clarity is gained by the time of the Reformation and the Enlightenment. First of all, there seems little doubt that the fundamental premise of "The Doctrine of the Timeaus" was wholeheartedly embraced by many scholars, who in turn nurtured it and labored to pass it on, dangers and troubled times notwithstanding. Secondly, in view of statements in ancient, medieval and later sources, it begins to becomes apparent, to some extent at least, what was being handed down. Taken together, the implications of "organic motion" introduced by Archytas, reference by Proclus to "physiology," the scope of the Pythagorean Tetractys, the texts, canons and aphorisms of Alchemists, and the bald reference to Phyllotaxis by Peirce and Agassiz all point to an active, extensive view of life itself, and ultimately, one might suggest, something akin to a living universe as the ancients perceived it.
Whether comprehension of an all-inclusive, all-pervading living universe is in line with the scale and the intent of Plato's Timaeus as Proclus understood it is likely to remain uncertain. But either way, given the turmoil and strife in the Middle Ages and long thereafter, it is understandable that this all-encompassing viewpoint could not become generally known or widely propagated. The intent, however, may still have flourished, albeit guardedly, and from this viewpoint, it might well be that Sir Isaac Newton, renowned scientist but also an alchemist, was correct in his day when he offered the following conclusion to his Mathematical Principles of Natural Philosophy: ${ }^{97}$

[^0]To what degree this assessment implies the acceptance of a complex, living universe is also unknown, but in view of the massive outpouring of obscure yet highly detailed literature from the Alchemists before and after Newton's time, it can be surmised that the Enlightenment was disappointingly incomplete without this all-encompassing viewpoint. As for this failure, if that is what it was, this is not a matter for recrimination or blame, but in retrospect it was in all likelihood simply the wrong time and the wrong place.

As for the present, one would like to think that we are more than ready now. In fact four centuries have elapsed since Johannes Kepler and the year 1618. So, though long overdue, perhaps the Enlightenment can continue and the concept of the living universe, or whatever name is appropriate, be permitted to run unfettered alongside both Science and Religion.

Though resistance to change is almost universal, for religions the concept of God is generally synonymous with an all-pervading entity, thus the living universe is essentially one more name for the Almighty according to numerous cultural perspectives. Science, on the other hand, will still retain the "Big Bang Theory" as a primary hypothesis, possibly influenced by "living" aspects of the matter, and quite possibly not. But either way, one might also hope this awareness may eventually help to establish common ground between science, religion and atheism, and all persuasions and passions in between.

In any event, in these troubled times, what might be needed above all else is a sense of belonging accompanied by a related sense of responsibility towards others, all forms of life, and not least of all, our immediate environment. As for myself, these fundamental issues are aptly indexed by NUMEN LUMEN, the motto of the University of Wisconsin in Madison, USA where this inquiry began in the darkness before the dawn more than fifty wandering years ago:

## The Divine within the Universe, however manifested, is my Light.

And possibly, the Light that fills the World ...

Only On s Great Thing

It think over again My Small Adventures,

$$
\text { My } \mathcal{F}_{\text {gars. }}
$$

Thous Small ones that seemed so Big.
Of all the Vital Things
1 Had to $\Theta_{\text {st }}$ and Peach.
And yest there is only $O_{n \varepsilon}$ Great Thing, $^{\text {Treat }}$
The Only Thing,
Jo Live to 1 re the Great Day that Dawns, And the Light that fills the World.*

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## PART ONE



## EXCURSUS

HISTORICAL ISSUES \& PROBLEMS
(OPTIONAL)

## I. The Babylonian Ecliptic Reference Frame and the Zodiac

Apart from the relatively recent Astronomical Diaries and Related Texts by Abraham J. Sachs and Hermann Hunger (1988) this is still a neglected area. Yet there is literally nothing in the historical record which comes remotely close to details in mathematical cuneiform texts from the Old Babylonian Period (1900 BCE-1650 BCE), the Ephemerides, the various "Procedure" texts of the Seleucid Era (310 BCE - 75 CE) and the Babylonian Astronomical Diaries from 625 BCE - 65 BCE. Included among the latter are the two frames of reference against which the major astronomical phenomena take place; 1. The local horizon, and 2. A fixed celestial reference frame provided by +33 bright stars distributed around the ecliptic, i.e., with in a few degrees of the paths of the visible planets.

The Diaries represent a prime resource while also including background information on the subject plus three sets of longitudes (L) and inclinations (i) of the Babylonian ecliptic reference stars for 600 BCE, 300 BCE and the Year 0 CE. For more than enough data, in fact, to display the zodiacal reference frame and the ecliptic "normal" stars with their various names and signs against those published in Neugebauer's Astronomical Cuneiform Texts (1955) to overcome the difficulties caused by the latter's overly condensed and far from helpful presentations of both the texts and their contents. Thus, starting with Figure 2a below, the dual names and signs for the Zodiac used in this major reference followed by a condensed presentation of Babylonian planetary parameters and periods relations on the next page.


Fig. 2a. Babylonian ecliptic "Normal" Stars; the Zodiac plus standard signs and dual names

| $\gamma$ Aries (ḥun) | $0^{\circ}$ | ठ Leo | (a) | $120^{\circ}$ | $\theta$ Sagittarius | (Pa) | $240^{\circ}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Y Taurus (múl) | $30^{\circ}$ | mp Virgo | (absin ${ }_{0}$ ) | $150^{\circ}$ | If Capricorn | (máṡ) | $270^{\circ}$ |
| II Gemini (máṡ-más) | $60^{\circ}$ | $\simeq$ Libra | (rin) | $180^{\circ}$ | \% Aquarius | (gu) | $300^{\circ}$ |
| ๑ Cancer (kusú) | $90^{\circ}$ | m Scorpio | (gir-tab) | $210^{\circ}$ | )( Pisces | (zib) | $330^{\circ}$ |

## I. BABYLONIAN PLANETARY PARAMETERS AND PERIOD RELATIONS

The five planets known in Antiquity are shown in planview north of the ecliptic in heliocentric order with Earth (following Mercury and Venus) third planet from the Sun. Also supplied are modern estimates for the mean periods of revolution and calculated mean synodic periods/lap-cycles from relations (2s) and (2i) given on page 2.
A. Mean Periods of revolution $(T)^{6}$

|  | PLANET <br> Sidereal | Modern T revolutions ${ }^{2}$ | Babylonian revolutions | Babylonian <br> Ratio: N/Z | $\begin{aligned} & \text { Diff. (T) } \\ & \text { \% Diff. } \end{aligned}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 3 | Saturn | 29.423519 | 29.444444 | (265/9) | -2.093\% |
| 4 | Jupiter | 11.856520 | 11.861111 | (427/36) | -0.459\% |
| 5 | [M-J Gap] |  |  |  | - |
| 6 | Mars | 1.8807111 | 1.8807947 | (284/151) | 0.0084\% |
| - | [ Earth] | 1 | 1 | [1] | - |
| 7 | Venus | 0.6151857 | $0.6151790^{2}$ | (1151/1871) | 0.0004\% |
| 8 | Mercury | 0.2408444 | $0.2408377^{2}$ | (46/191) | 0.0007\% |



Table 4. Modern and Babylonian mean periods $T, S$ in years
Fig.3. The Orbits of the five planets known in Antiquity
Synodic relations (1), (2) and the modern equivalents all have roles to play in Table 4, but relation (1) in full has a further application arising from the inclusion of the mean synodic month in Babylonian planetary theory beyond mere calendaric considerations. More on this later.

Babylonian mean periods, on the other hand, are from data-sets which provide insights into their determination, initial lack of revolutions for the two inferior planets included. As they now stand and as presented in ACT (1955) by Neugebauer, Babylonian mean periods were, however, seemingly "simple" integer period relations in a dual form: ${ }^{28}$

$$
\begin{array}{ll}
\text { Superior planets: } N \text { years : II Synodic arcs/periods : Z Rotations; } & \text { N Years = II Synodic arcs + Z Rotations } \\
\text { Inferior planets: } & \text { N years : Il Synodic arcs/periods. (only) }
\end{array}
$$

whereas, in contrast to Neugebauer's treatment, from a fictive, heliocentric viewpoint and revolutions substituted for"rotations" plus the mean periods of revolution $(Z)$ added for Venus and Mercury, a more complete and balanced arrangement is:

> THE SUPERIOR PLANETS: SATURN, JUPITER AND MARS
> SATURN: 265 years -256 Synodic arcs $=9$ Revolutions
> JUPITER: 427 years -391 Synodic arcs $=36$ Revolutions
> MARS: $\quad 284$ years -133 Synodic arcs $=151$ Revolutions
> THE INFERIOR PLANETS: VENUS AND MERCURY
> VENUS: 1151 years +720 Synodic arcs $=1871$ Revolutions (added)
> MERCURY: 46 years +145 Synodic arcs $=191$ Revolutions (added)
> Also ( shown later) four more period relations for Mercury are attested.
which yields by integer division results which are comparable to the modern values shown in Table 4.

## II. Babylonian "characteristic" synodic phenomena

From a modern viewpoint it is easiest to start with Jupiter at Opposition to introduce the "characteristic" synodic phenomena since of the five Babylonian synodic events it is only at Opposition that the location of Jupiter is at its true position. Whereas, for three of the other four phenomena the locations are merely apparent, while for the fourth (from the Last Visibility in the West to the First Visibility in the East) Jupiter is moving on the opposite side of the Solar System and cannot be observed at all throughout the interval. For this reason the following figures include the annual sidereal motion for Jupiter with Figure 4a initially assigned an integer value of 12 years for the period of revolution $(T)$ before including the Babylonian characteristic phenomena as an integral set in Figure 4b.


Fig. 4a. The mean synodic arc $u$ for Jupiter, theoretical $T_{1} 12$ years. Babylonian $T_{1}$ as used $11.86111^{*}$ years.


Fig. 4b. Jupiter: Characteristic Phenomena and concurrent heliocentric motion of Earth.

Shown in planview north of the ecliptic in Figure 6, the Babylonian reference frames enclose the "characteristic" synodic phenomena for Jupiter on the $34^{\circ}$ synodic arc accompanied by the direct heliocentric motion of Earth. As perhaps stated in Inset A and Note 1: ${ }^{29}$


Fig 5. The Zodiac, Babylonian Ecliptic Reference Stars, Synodic Phenomena, Heliocentric motion of Earth.
${ }^{1 .}$ ACT 817 , Section 2, Line 8 in full: "Dr. Sachs suggested the following interpretation of line $8:{ }^{\prime}$ Jupiter. At one-third of its (stretch
of) visibility: first station; at one-half of its (stretch of visibility: [Opposition; at two-thirds: second station]. This is indeed in good
agreement with the general schemes for subdividing the synodic arc of Jupiter."
Otto Neugebauer, ACT (1955: 430).

But It is EARTH which is moving approximately "one-third", then "one half", followed by "two-thirds" over the full synodic arc of $360^{\circ}+u$. Plus the period for Earth to move from the Last visibility of Jupiter in the West $\left(\Omega_{1}\right)$ to the First visibility of Jupiter in the East ( $\Gamma$ ) from ACT 813, Section 11 is 27 days (in Cancer $\circlearrowleft$ ) and 32 days (in Sagittarius $\psi$ ). 30 Thus a mean value of 29.5 days for the varying intervals of invisibility versus the mean synodic month ( 29.5306 days) with regular variations arising from the elliptical orbits of Moon, Earth and Jupiter. Plus considerable problems from parallax to further complicate the precise location of Jupiter on the day of the First visibility in the East.


Fig. 6. Heliocentric motions of Jupiter and Earth from line 8, Section 2 of ACT 817.

## III. The determination of the Babylonian Integer Period Relations

Shown in plan view north of the ecliptic it has long been recognised that the relatively small eccentricities of the Solar System planets cause the orbits to resemble slightly off-set circles as opposed to elliptical orbits per se.
What might all this have to do with Babylonian planetary theory? A great deal, as it turns out, because - as in the case of the mean periods - modern eccentricities also have an a priori element to them. In other words, the mean distances and the eccentricities are given with the apsidal distances calculated from this base with respect to an already assigned line of apsides. Thus the locations of the shortest (and fastest moving) radius vectors/distances and the longest (also slowest moving) radius vectors for each planet. Whereas the Babylonians, having determined the mean periods and associated mean velocities from a lengthy series of observations, next seem to have derived the values for not only for the extremal synodic arcs, but also precise locations for the same. In short, lines of apsides. But first, how were the integer period relations which provided the Babylonian periods ( $T, S$ ) in Table 4a obtained ?

Fortunately, the origin of the integer relations is generally well known, even if the impressive results are not. Or perhaps better stated, still largely driven into obscurity by partial and less than praiseworthy modern analyses that fail to mention the accuracy so obtained. Nor do repeated references to "rotations" help here either, or an inability to consider the significance of the difference in format for the final relations of the Superior and Inferior planets.

## Babylonian "long" period relations

## The Final period relation for Jupiter: In 427 years: 391 synodic arcs and 36 Revolutions

The method for obtaining the mean and varying velocities and the associated times is explained in great detail in the procedure texts from the Seleucid Era, with those for Jupiter among the most informative, e.g., as in ACT 813, a Babylonian "procedure" text for the determination of the mean synodic arc which in Sections 1 and 21 includes the simple integer period relation for Jupiter of 427 years as the end product: ${ }^{31}$

[^1]The translation and reference to the Almagest are by Neugebauer; decimal periods have been added for clarity and "revolutions" have also been substituted for "rotations" uniformly promoted by the latter. In passing it is safe to say that the latter replacement(s) largely correct Neugebauer's generally inadequate terminology.

## Origins and Implications

It is apparent from this procedure that for Jupiter a pair of integer periods were selected (named here $T 1$ and $T 2$ ) with corresponding near-integer numbers of synodic periods, near-integer numbers of revolutions with small and convenient corrections in longitude of opposite sign. From these periods and corrections the final Babylonian "long" period integer relation of 427 years is readily obtained by addition. As in fact suggested by the writer in a Letter to the Editor of ISIS published in $1977{ }^{32}$ in an earlier attempt to close Babylonian orbits about the common center.

| T | Periods (Years) | Progress T1-T7 | Synodic Synodic arcs | Sidereal Revolutions | Corrections (Longitude) |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| T1 | 12 | (12) | 11 | $1 \times 360^{\circ}$ | $\begin{gathered} \text { Slow } \\ +4 ; 10^{\circ} \end{gathered}$ | $\begin{gathered} \text { Fast } \\ +5 ; 00^{\circ} \end{gathered}$ |
| T2 | 71 years | (6*12) | 65 | $6 \times 360^{\circ}$ | -5;00 ${ }^{\circ}$ | +6;00 ${ }^{\circ}$ |
| T3 | 83 years | $(12+71)$ | 76 | $7 \times 360^{\circ}$ | - 0;50 ${ }^{\circ}$ | $-1 ; 00^{\circ}$ |
| T4 | 95 years | $(12+83)$ | 87 | $8 \times 360^{\circ}$ | $+3 ; 20^{\circ}$ | $+4 ; 00^{\circ}$ |
| T5 | 166 years | $(71+95)$ | 152 | $14 \times 360^{\circ}$ | -1;40 ${ }^{\circ}$ | -2;00 |
| T6 | 261 years | $(166+95)$ | 239 | $22 \times 360^{\circ}$ | +1;40 | +2;00 ${ }^{\circ}$ |
| F7 | 427 years | ( $166+261$ ) | 391 | $36 \times 360^{\circ}$ | 0;00 ${ }^{\circ}$ | 0;00 |

Table 5. Babylonian Jupiter corrections and the final 427-year period $F$.
Although Neugebauer was aware of the corrections and almost all of the periods which lead to F7, the final integer period relation, he did not pursue the matter further, although (on inspection) closed orbits are obviously involved since each correction is also heliocentric in nature. For example, by the same process used for the synodic arc, the annual sidereal arc for Jupiter is obtained from the division of the total sidereal motion (as before $36 \cdot 360^{\circ}=12,960^{\circ}$ ) by 427, the number of years for $\boldsymbol{F}$, the final period. The mean sidereal motion for Jupiter ( $30.351288056^{\circ}$ per annum) applied to the $T 1$ and $T 2$ periods gives the excess over 12 years ( $T 1$ ) and the deficiency after 71 years ( $T 2$ ) modulo $360^{\circ}$ which results in values of $4.2152457^{\circ}$ and $-5.058548^{\circ}(4 ; 12,55,38$, and $-5 ; 3,30,46$,$) thus rounded to the nearest$ sixth of a degree $\left(0 ; 10^{\circ}\right)$ for both, values of $4 ; 10^{\circ}$ and $-5 ; 00^{\circ}$ as used by the Babylonians.

## The 427-year Final period for Jupiter

Clearly, from the above the initial stage of the Babylonian inquiry appears to have been the derivation of an integer number of years to which there corresponded an integer number of mean revolutions and a corresponding integer number of mean synodic periods. With this primary goal for Jupiter achieved, the next step following in Section 21 of ACT 813 is the determination of the mean synodic arc $(u)$ as follows: ${ }^{33}$
[ 7,7 years (correspond to) 6,31 appearances (),] 36 rotations, 3,36, $0^{\circ}$ motion, $33 ; 8,[4] 5$ (is the) mean value of the longitudes.
which in plain English and decimal notation is clearly the unambiguous statement that:
427 years correspond to 391 synodic periods, 36 revolutions, $12,960^{\circ}$ sidereal motion, $33 ; 8,45$ (is the) mean value of the synodic arc. (added: with $33 ; 8,44,48,29,{ }^{\circ} . \ldots$. . rounded to $33 ; 8,45^{\circ}$ ).

Thus, unequivocally, 36 revolutions of mean circular motion with a subdivided arc subtended from the center with 427 years also the 427 revolutions of Earth. And as seen next, the same argument is applicable to the division of the total sidereal motion of Saturn by the corresponding number of mean synodic periods to derive the mean synodic arc for this planet in the same manner. Except here the result is exact and convenient in both base-60 and base-10.

## The 265-year Final period relation for Saturn

The identical procedure is given for Saturn in abbreviated form with the total sidereal motion supplied in Section 4 of ACT 802, a Saturn procedure text. Which in decimals with revolutions substituted for "rotations," is: ${ }^{34}$

Concerning Saturn. 265 years corresponds to 256 synodic periods, 9 sidereal revolutions and 3,240 total motion.
leading to the mean synodic arc for Saturn $u\left(12 ; 39,22,30^{\circ}\right)$ from the total sidereal motion $\left(9 \bullet 360^{\circ}=3,240^{\circ}\right)$ divided by the number of mean synodic periods (256). Section 5 supplies the constants for the extremal synodic periods, and Section 6 the corresponding extremal synodic arcs. The two intermediate periods $T 1$ and $T 2$ with corrections of opposite sign for Saturn are 29 years and 59 years which lead in turn to the 265 -year integer relation $F$ of Saturn.

## The 284-year Final period relation for Mars

For Mars $T 1=47$ years, $T 2=79$ years with the Final integer period $F_{\mathrm{n}}=284$ years, to which correspond $151 \times\left(360^{\circ}\right)$ periods of revolution and 133 mean synodic arcs/mean synodic periods. The total motion is again divided by the number of synodic events $\left(151 \cdot 360^{\circ}=54,360^{\circ}\right)$ divided by $133=408 ; 43,18,29,46, \ldots$ abbreviated to $408 ; 43,18,30^{\circ}$ and ultimately $u=48 ; 43,18,30^{\circ}$. This differs markedly from Jupiter and Saturn where Earth moves $\left(360^{\circ}+u\right)$ in the time that it takes for these two outer planets to move the mean synodic arc $u$ alone. Now, to lap Mars, Earth must move $\left(2 \cdot 360^{\circ}+u\right)$ during which time Mars itself moves $\left(1 \cdot 360^{\circ}+u\right)$. This departure from the format for both Saturn and Jupiter is perhaps one reason for earlier uncertainty concerning any Babylonian System B velocity function for Mars despite hints in Mars procedure text ACT 811a. ${ }^{35}$

## IV. The Inferior planets Venus and Mercury. Venus

In addition to the well-known but barely recognized Fibonacci pairing of 5 synodic periods/arcs in 8 years, a longer Final period of 720 synodic periods/arcs in 1151 years is also attested. The number of revolutions for this planet are again absent, with the final integer relation merely the numbers of years and corresponding numbers of synodic arcs/periods: ${ }^{36}$

Venus: 8 years corresponds to 5 synodic arcs/periods
Venus: 1151 years corresponds to 720 synodic arcs/periods
with the ratio of the two parameters providing the mean synodic period(s) in years: $8 / 5=1.6$, and more accurate $1151 / 720=1.5986111^{*}$ years. Expressed in days the latter mean synodic period for Venus is $583.9097^{d}$ versus the modern estimate of $583.9214^{\text {d }}$.

## Mercury

Five Babylonian period relations for Mercury are attested with 145 synodic periods in 46 year the best known. ${ }^{37}$ As in the case for Venus the corresponding number of revolutions is not stated, an apparent deficiency maintained by Claudius Ptolemy in the Almagest and a mistake also retained by almost all subsequent commentators.

The resulting mean synodic period for Mercury is quite accurate, i.e., $46 / 145=0.317241$ years and 115.8758 days versus the modern estimate of 115.8775 days. The same degree of accuracy is also provided by the four remaining integer period relations. The latter set of limited Babylonian period relations for Mercury replete with Greek-letter assignments provided by Neugebauer are included below with the missing numbers of sidereal revolutions and the missing revolutions $(Z=T)$ supplied. ${ }^{38}$ The period of revolution ( $T$ ) for the 46 -year, 145 synodic period relation (and corresponding 191 revolutions) is 0.24083769 years ; the modern estimate for $T$ is 0.24084445 years.

> In 388 years, 1223 disappearances as a morning star (and 1611 revolutions, $T=0.24084419$ years) In 480 years, 1513 appearances as an evening star (and 1993 revolutions, $T=0.24084295$ years) In 217 years, 684 disappearances as an evening star(and 901 revolutions, $T=0.24084350$ years) In 848 years, 2673 appearances as a morning star ( and 3521 revolutions, $T=0.24084067$ years)

Thus the 46-year period relation for Mercury does not even provide the best result, nor in their original Babylonian context do any of the integer period relations require a "correction," despite the later "improvements" by Claudius Ptolemy carried forward with the latter's defective geocentric planetary model to the time of Copernicus.

## VI. The significance of the unstated periods of revolution for the Inferior planets

Although the final period (F) differs in format for the two Inferior planets from those applied for Mars, Jupiter and and Saturn, the lack of stipulated numbers of periods of revolution for Venus and Mercury is nonetheless merely a function of the heliocentric nature of Babylonian planetary theory. In other words, though the motion of Earth provides the frames of reference for both the times and the velocities, it is always the outer planet of the respective synodic pair that provides the synodic arc. For the superior planets it is Earth which laps Mars, Jupiter and Saturn during each synodic period, i.e., each lap-cycle:

## SUPERIOR PLANETS

From the Number of Years, corresponding numbers of Synodic arcs/periods and numbers of Revolutions Outer and Slower-moving SATURN, and JUPITER move the direct (synodic) arc ( $u$ ). Inner and Faster-moving EARTH concurrently moves $\left(360^{\circ}+u\right)$ to supply the time per lap-cycle. Mars is the exception where Earth moves $\left(2 \bullet 360^{\circ}+u\right)$ during which time Mars moves $\left(1 \cdot 360^{\circ}+u\right)$.

1. EARTH in all cases supplies the mean synodic time per lap-cycle, but not the mean synodic arc (u).

## INFERIOR PLANETS

From the relation: Number of Years and corresponding number of Synodic arcs/periods only Outer and Slower-moving EARTH moves the direct (synodic) arc (u).
while inner and faster-moving VENUS and MERCURY concurrently move ( $n \cdot 360^{\circ}+u$ ) per lap-cycle.

## 1. EARTH supplies the mean synodic time per lap-cycle. 2. EARTH also supplies the mean synodic arc (u).

VENUS: In 1151 years, 720 synodic cycles (and unstated: 1871 revolutions)
Mean Synodic time: $1151 / 720=1.5986111^{*}(1 ; 35,55)$ years, or 583.909714 days. Mean synodic arc $(u):\left(1151 \cdot 360^{\circ} / 720\right)=575.5^{\circ}$ and/or $215.5^{\circ}$ from $\left(1 \cdot 360^{\circ}+u\right)$. during which time VENUS concurrently moves $\left(2 \cdot 360^{\circ}+u\right)$.

MERCURY: In 46 years, 145 synodic cycles (and unstated: 191 revolutions)
Mean Synodic time: $46 / 145=0.3172413793$ years or 115.8757885 days. Mean synodic arc (u): $\left(46 \cdot 360^{\circ} / 145\right), u=114.2068965^{\circ}$. during which time Mercury concurrently moves $\left(1 \cdot 360^{\circ}+u\right)$.
Thus from the heliocentric viewpoint, for the inner planets Earth is now supplying the synodic arc as an outer planet, but still providing the synodic time as before. Thus a variant of the known relation for the Superior planets where the number of revolutions are components of the integer relation itself. Therefore, remaining with the terminology and the symbols used by Neugebauer (albeit with revolutions substituted for rotations), the primary Superior planet relation necessarily includes the following re-arrangement of terms:

## SUPERIOR PLANETS. Variant of the primary period relation for Saturn, Jupiter and Mars:

Number of Sidereal Revolutions (Z) = Number of Years (N) - Number of Synodic periods (II)
e.g., returning to Jupiter, 36 revolutions = (427 years-391 synodic periods) with the mean synodic arc $(u)=33 ; 8,45^{\circ}$ derived and rounded as given in ACT 813, Section 21. Although not needed for practical reasons, should the actual periods of revolution for Venus or Mercury be required they can be readily obtained from the following variant:

## INFERIOR PLANETS. Variant of the primary period relation

Number of Sidereal Revolutions (Z) = Number of Synodic periods (II) + Number of Years (N)
Thus for Venus the Fibonacci series itself, i.e.,
5 Synodic periods +8 Years $=13$ sidereal revolutions ( $5+8+13$, etc.),
with the mean period of revolution ( $Z$ ) for Venus from: $8 / 13=0.615385$ years, versus:

1. The modern estimate $=0.615186$ years
2. Venus from $1151 / 1871=0.615179$ years
3. The reciprocal of $P h i=0.618034$ years

## VII. Babylonian astronomy: a modern enigma

Lastly, although the astronomical assignments for the Old Babylonian Period in Text-Fig 1are at odds with prevailing chronology and understanding they are not made lightly or without supporting research and textual support.

Nevertheless, as far as Babylonian astronomy is concerned, primary modern works by Otto Neugebauer (1955, ${ }^{39}$ $1975{ }^{40}$ ), Bartel van der Waerden (1974) ${ }^{41}$ and Noel Swerdlow (1998) ${ }^{42}$ accept the odd premise - despite procedures and details in the astronomical cuneiform texts - that the Babylonians possessed no fictive planetary model at all.

In fact, by the time of his final opus, A History of Ancient Mathematical Astronomy (1975) Neugebauer states with ill-founded certitude that the understanding of Babylonian astronomy : ${ }^{43}$

> is a rather self-contained problem of great technical complexity that will occupy us in the following chapters. But it will reveal to us the working of a theoretical astronomy which operates without any model of a spherical universe, without circular motions and all the other concepts which seemed "a priori " necessary for the investigation of celestial phenomena. The influence of the available mathematical tools on the development of a scientific theory is perhaps far greater than we customarily admit. Otto Neugebauer, A History of Mathematical Astronomy (1975:347-348; italics supplied)

Unfortunately, this nihilistic, pre-conclusive viewpoint in consort with base-60 mathematics, unfamiliar procedures, hard to follow names and concepts has rendered the Babylonian material difficult to read in the first place and not worth reading in the second. Hence the unwarranted yet still prevailing ignorance on the matter.

Quite literally, Neugebauer "wrote the book on the subject," i.e., Astronomical Cuneiform Texts published in1955, but this last assessment is not only erroneous, it also leaves one wondering how Neugebauer managed to escape critical peer reviews, the opposing discussion of precession and Babylonian years by W. Hartner (1979) ${ }^{26}$ excepted. Yet according to Neugebauer Babylonian astronomy exists without "any model" and "without circular motions and all other concepts . . . . necessary for the investigation of celestial phenomena," which is readily countered by the

Babylonian methods for the determination of fundamental period relations and mean synodic arcs provided in the various "procedure" texts of the Seleucid Era. Except that this material - replete with commentaries and notes - is in Neugebauer's Astronomical Cuneiform Texts (1955), and furthermore, key Babylonian procedures in this initial work either remain erroneous or incomplete. Whereas a further impediment to understanding - also introduced by the latter in his latest compendium - was a far from helpful caution, namely that: ${ }^{44}$

> In working with sexagesimally written texts it is essential not to convert the numbers to decimals, to carry out operations decimally and only to change results back to sexagesimals ... roundings in one system do not mean the same in the other and accurate parameters given in sexagesimals may be altered by the transition through decimal computations.
> Otto Neugebauer, A History of Mathematical Astronomy (1975:1113)

Thus Neugebauer's unfathomable rejection of circular motion and further insistence on sexagesimal mathematics for Babylonian astronomy. But his earlier closing remark concerning mathematical tools was not only incorrect, it was also decidedly unwise. In fact, it is the height of irony that around the time HAMA was published (1975), Robert R. Newton, using a 10-decimal place pocket calculator put an end to the entirely unmerited reign of the Almagest by showing that the latter's supposedly accurate daily velocities for the planets (each given by Ptolemy to the sixth sexagesimal place) were all erroneous. ${ }^{45}$ And therefore, they no longer "saved the phenomena," as claimed, while Otto Neugebauer, having spent his illustrious career heaping praise on Claudius Ptolemy and the Almagest, was obliged to admit in a feeble footnote in HAMA that he too had never checked the accuracy of the daily velocities in the chronically flawed Almagest. ${ }^{46}$ More to the immediate point, it also turns out that the velocities so used were in all likelihood appropriated from limited, earlier sets of completely misunderstood Babylonian period relations.

It could be said that awareness of these errors only came to light with the publication of Robert R. Newton's The Crime of Claudius Ptolemy (1977) ${ }^{45}$ and his The Origins of Ptolemy's Astronomical Parameters (1982), ${ }^{47}$ but this is not entirely so. Both AI-Bitruji (ca. 1200) ${ }^{48}$ and Copernicus (ca.1500), ${ }^{49}$ using for the most part the same period relations as Ptolemy were also unable to obtain compatible results, although the former, with a liberal sprinkling of the word "about," at least forewarns the reader of the uncertainties involved.

The main difficulty, it would seem, is not so much geocentricity, but restrictions imposed by having to incorporate uniform circular motion in association with basically unworkable auxiliary devices. Plus one further restriction - also never met with Ptolemy's values - which was that the sums of the supposed velocities in "anomaly" and "longitude" in each and every case were required to equal the velocity of the Sun.

One can certainly work though all the period relations employed by Ptolemy, Al-Bitruji and Copernicus to check the disparities discussed by Robert R. Newton, although it is nevertheless an irritating task. At present it is sufficient to note that as recently as the second half of the Twentieth Century - Manitius ${ }^{50}$ (1963 reprint), Pedersen (1974) ${ }^{51}$ and even Neugebauer (1975) ${ }^{52}$ were unaware of the multiple errors in the daily velocities in Ptolemy's Almagest.

Perhaps Neugebauer was encouraged and emboldened by lack of criticism for earlier omissions in ACT (1955), not to mention its unreadable and disjointed presentation, but for whatever the reasons, even passing knowledge of the Babylonian procedures in the astronomical cuneiform texts of the Seleucid Era - the primary source for both ACT(1955) and HAMA (1975) - indicates that the Babylonian methodology routinely and indisputably employed uniform circular motion for the initial derivation of the mean synodic arcs for Sun, Moon and the planets. But unlike the Ptolemaic planetary model which utilized uniform circular motion and auxiliary devices to account for observed variations, the Babylonian used mean circular motion for the mean synodic arcs as the basis for schemes to account for variations on either side of established mean values. Circular motion is certainly applied in Babylonian astronomy but as an intermediate step towards the determination of apsidal velocities (predominantly System A) with more detailed schemes to determine the varying velocities and the associated varying times in System B. All of which was achieved before the time of Hipparchus (ca. 150 BCE ) and well before Claudius Ptolemy's much-praised Almagest. Furthermore, the same fractional parts of misunderstood Babylonian period relations were, it would seem, not only applied by the latter, but also used in the alternate geocentric planetary model by Al-Bitruji (ca.1200) and ultimately in the heliocentric model proposed by Nicholas Copernicus (ca.1500).

## VIII. Babylonian astronomy during the 20th Century

Then how did the current mindset which holds that the Babylonians themselves possessed no planetary model or any fictive understanding become established? There are numerous issues in ACT for which corrections are long overdue with unknown parameters and operations in the Babylonian procedure texts in need of explanation. Some deficiencies are elementary enough, but they should surely have been fully addressed by now, specially for Jupiter. This is clear from the distribution of the planetary texts in ACT cataloged by Neugebauer in his Introduction to the Planetary Theory in General: ${ }^{53}$

Our material contains ephemerides for all five planets. The distribution is very uneven: 41 texts and fragments for Jupiter, but only 12 for Saturn, 11 for Mercury, 9 for Venus, and 8 for Mars. In quoting these texts, we adopt the the following notation:

$$
\begin{array}{ll}
\text { Nos. } 300 \mathrm{ff} . & \text { Mercury } \\
\text { Nos. } 400 \mathrm{ff} . & \text { Venus } \\
\text { Nos. } 500 \mathrm{ff} . & \text { Mars } \\
\text { Nos. } 600 \mathrm{ff.} & \text { Jupiter } \\
\text { Nos. } 700 \mathrm{ff.} & \text { Saturn } \\
\text { Nos. } 800 \mathrm{ff} . & \text { Procedure texts }
\end{array}
$$

The "procedure texts" also deal with all five planets but here, too, Jupiter is better represented than all the other planets together.
The foundation of our understanding of the planetary texts was laid by Kugler' and Pannekeok. ${ }^{2}$ Later contributions were due to Schnabel, ${ }^{3}$ and van der Waerden. ${ }^{4}$ Our material seems to be large enough to justify the statement that the planetary theory was less developed than the theory of the moon. We find, e.g., no direct consideration of the latitude. The complications of the planetary motion on one hand, the lack of of calendaric importance on the other might be the reasons for this fact.

It must not be forgotten, however, that our knowledge of Babylonian planetary theory rests on a much smaller basis than the lunar theory. Not only is the total number of texts for all five planets less than half of the lunar texts, but also the state of preservation of the procedure texts is much worse in the case of the planets than for the moon.

The inclusion of"procedure" texts is an initial surprise, as is the larger luni-solar component with its potential for providing insights concerning the Babylonian approach to Earth/Sun dualities. This is the essential matter that lies at the heart and soul of all Solar System planetary theories.

So far Neugebauer's introduction to the Babylonian material is encouraging as well as intriguing, and the same can be said concerning the cautions included in his explanation of "The Leading Ideas of the Planetary Theory." ${ }^{54}$

> The discussion of the Babylonian planetary theory which follows is not intended to give a historically or astronomic -ally complete description of this field. Its only purpose is to prepare the reader for the study of the texts themselves, that is, to enable him to compute the ephemerides with the methods used used by the ancients and to grasp the general trend in the procedure texts. For all details, however, he must consult the texts themselves plus the notes and commentaries added to the translations and the relevant remarks in the Introductions concerning the theory of the single planets. The reader must remember what I have said in the preface: this is only an edition of the texts, not a systematic study of the contents.
> Otto Neugebauer, Astronomical Cuneiform Texts (1955:279)

But if ACT (1955) was indeed "only an edition of the texts, not a systematic study of the contents," then without further analysis, how did the notion that this material was insufficient to support a fictive planetary model arise? A partial answer would seem to be that prior analytics with initial emphases on available tools - units of time and measure, astronomical reference frames, mathematical methodology and results as recorded in the astronomical cuneiform texts - were not carried out adequately. Yet the latter seem to be relatively limited sets of data, i.e., the cumulative progress of the planets in terms of varying synodic arcs with accompanying synodic times, in short, to ${ }^{55}$

> find the longitudes and dates for the consecutive "characteristic phenomena" of the planetary appearances, such as helical risings and settings, oppositions, stations, etc. Only if these phenomena are known, can the location of the planet be determined for intermediate moments by means of complicated interpolations. But the overwhelming majority of the "ephemerides" and the procedure texts are devoted to the problems of determining the sequence of the characteristic phenomena, whereas the problem of "daily motion" is of a secondary character.
> Otto Neugebauer, Astronomical Cuneiform Texts (1955:279)

The "secondary" nature of daily motion is not a problem if it is given due consideration. It does, moreover, supply a reminder that the diurnal axial rotation of Earth necessarily plays a fundamental role in the various "risings," "settings," "disappearances/last visibilities in the West" and "appearances/first visibilities in the East" of the synodic phenomena. Not alone, however, but with the daily rotation of Earth about its axis while simultaneously completing its annual orbital revolution about a common center, and in the same way the longitudinal progress of each planet. But it is here that Neugebauer's description of Babylonian synodic phenomena takes a decidedly strange turn. First of all, although he is willing to concede that the Babylonian approach "is obviously the 'natural ' one," he nevertheless follows this reasonable statement with an historical aside that negates it while elevating later Greek achievements. Which in the case of Ptolemy and the vaunted Almagest turns out to be quite unmerited. Thus in full Neugebauer asserts that: ${ }^{55}$

The Babylonian approach is obviously the'natural' one.' What one realizes first about the planets is their appearances and disappearances in the nightly sky, their stations and retrogradations. To predict these phenomena seems to be a real problem and it was solved by our texts by means of very ingenious arithmetical devices. But it marks an enormous step forward to ignore the'natural' problems altogether and to ask an apparently much more complex question: how to describe the planetary motion as a whole. It is this shift of emphasis which led Apollonius, Hipparchus, and Ptolemy to their enormous successes.

Unfortunately, most new readers would accept this assessment without question, coming as is does from noted Historian of Science Neugebauer who is also the foremost authority on Babylonian astronomy. Even so, from what he has said earlier about the limited and specialized nature of the Babylonian approach it would be premature to embark on such comparisons. But more to the point, despite Neugebauer's proclamations and denigrations, the instructions to his readers to consult the contents of the Babylonian texts provides the means to test whether the Babylonian methodology does (or does not) describe "planetary motion as a whole." And also - a necessary asidewhy would anyone conclude that it did not, with the form of their approach of secondary importance, even if it is almost self-evidently heliocentric ?

The main problem, it seems, is that Neugebauer not only treated the Babylonian materials in the wrong order, he also failed to provide the necessary groundwork and the requisite tools for the job. Furthermore, commencing with the relatively complex motions of the moon followed by a convoluted treatment of the motions of the two Inferior planets Venus and Mercury was premature in the absence of a coherent treatment of lunar and planetary motion. Which is not to make light of such requirements, even though matters are helped to a considerable extent by the application of essentially the same synodic formulas in both lunar and planetary contexts. Which is hardly surprising in view of the name allotted to the commonly observed period for the moon, i.e., the "synodic" month, simplistically, from one full moon to the next. But what is not simple, however, is that although there is only one moon orbiting Earth, its motions occur at the same time that the latter - while concurrently rotating daily about its axis - is also orbiting the Sun and completing one full revolution of $360^{\circ}$ in one year. And once again, there is more than one kind of month and more than one kind of year, not to mention additional inter-related cycles with periodic variations caused by the elliptical natures of the various orbits, inclinations and precession, etc. Plus the fact that the observing platform (Earth) is subject to further variations caused by the considerable tilt of its own rotational axis.

Lastly, it should be noted that so far the present inquiry has been limited to the mean values alone. Omitted so far, the Babylonian treatment of varying velocities encountered in both luni-solar and planetary contexts is far more complex. Again this is best understood in heliocentric terms, current negative views on the matter notwithstanding. So, although not wishing to "concatenate without abruption," (as Dr. Johnson was wont put it) it is necessary to now include the Babylonian varying velocities in their various forms plus matters of related interest. But how could this be realized in the face of Neugebauer's bald statement that his analysis of Babylonian astronomy: ${ }^{37}$

> will reveal to us the working of a theoretical astronomy which operates without any model of a spherical universe, without circular motions and all the other concepts which seemed "a priori " necessary for the investigation of celestial phenomena.

The answer is simple enough. Easily, based on applied Babylonian synodic data and details in the "procedure" texts for Jupiter and Saturn, with both also employing the afore-denied "circular motions," and orbital progress observed, recorded and applied in Babylonian Systems A and B, where further surprises await.

## IX. Babylonian Systems $A$ and $B$ synodic arcs and times; elliptical orbits

So far the discussion of the parameters of Systems A and B has been largely concentrated on the apsidal synodic arcs of System A, the mean synodic arcs of System B, an intermediate mean arc from System A' and the unattested theoretical mean values which resulted in Fibonacci ratios of $8 / 5,5 / 3,8 / 3,144 / 55,55 / 144$ and $144 / 89$. Yet there are unfortunately difficulties with System B as Aaboe's 1964 analysis of BM 37089 demonstrated which can be traced to a lack of awareness of the underlying formula for the determination of the extremal arcs of System B. But this itself is not the only issue since a secondary part of the problem also involves the role played by the mean synodic month ( $M S M=29 ; 31,50,8,20^{d}$ ) in Babylonian planetary theory. It is here, however, that the choice of the slightly too high (but convenient) sidereal year MYR of $12 ; 22,8$ mean synodic months (or $\operatorname{SYR}=365 ; 15,38,17,44,26,40^{d}$ ) becomes clear, especially when used with the $360^{\circ}$ Zodiac's $12 \times 30^{\circ}$ zodiacal months and "tithis," $i . e$. ., thirtieths of the mean synodic month. In other words, rather than determine the time required for Jupiter to move the mean and varying synodic arcs using the standard year in days and mean synodic months (SYR and MYR above) parts of the work were done using thirtieths, which is a major simplification for a base-60 numerical system. And with pre-selected parameters, also finite, which is where the rounding of the mean synodic arc for Jupiter of $(u)=33 ; 8,45^{\circ}$ (from $33 ; 8,44,48,29, .{ }^{\circ}$ ) unfortunately also complicates matters, at least for those unfamiliar with the methodology.
The reason for this situation lies in the fact that while Babylonian "procedure" text ACT No. 812 provides a method for the mean synodic time for Jupiter in the badly damaged first section and an alternate approach is given in the second, the two methods yield different answers. Or more precisely, results given to three and five sexagesimal places respectively - " $13 ; 30,27,46$ " in the first section ${ }^{56}$ and " $13 ; 30,27,46,16,40$ " deducible from the conclusion of the second. Now had this value been given in full in the latter it is safe to say that Neugebauer (who was left "completely in the
dark" by the three-place version in the first section) would almost certainly have realised that both parameters were the mean synodic period of Jupiter expressed in mean synodic months with three-place 13;30,27,46 MSM the value applied in the first section. ${ }^{57}$

An unfortunate consequence of the damaged state of Section1 of ACT 812 is that already condensed details are further obscured, but the mention of an alternate approach in Section 2 suggests that the first method was merely the application of the Babylonian integer period relation for Jupiter, essentially, where MYR $=12 ; 22,8$ MSM:

36 revolutions (Z) and 391 synodic periods or arcs (II) in 427 (N) years
with the time for the mean synodic cycle simply $427 / 391 \cdot$ MYR $=13 ; 30,27,45,52, \ldots$ MSM necessarily rounded up at the fourth sexagesimal place to $13 ; 30,27,46$ MSM ( 398.89078921 . . days). Plus, by the same methodology and the integer long period relation for Saturn:

$$
\begin{equation*}
9 \text { revolutions (Z) and } 256 \text { synodic periods or arcs (II) in } 265 \text { (N) years } \tag{5}
\end{equation*}
$$

the mean synodic time for Saturn is: $265 / 256 \cdot$ MSM $=12 ; 48,13,26,15$ MSM ( 378.10183198 days) with both methods producing identical answers because no rounding is required for the mean synodic arc of Saturn ( $u=12 ; 39,22,30^{\circ}$ ). From this viewpoint the duality is best demonstrated by the Saturn integer period relation, but there is also a more critical matter involving the first section of ACT 812 to be addressed, which is the determination of the extremal or apsidal synodic arcs from the integer period relations. Which, despite the damaged state of Section 1 , is readily_ apparent from what remains. Neugebauer's transcription and commentary of No. 812 Section 1 plus one addition to line 6 is given below. The 15 pa in line 5 is the location of the mean value ( $u$ ) at15 Sagittarius in the Zodiac; 15 zib and 15 absin line 6 are the locations of the maximum ( $M$ ) and minimum ( $m$ ) arcs/times at 15 Pisces and 15 Virgo. ${ }^{58}$

$$
\begin{gathered}
\text { No. } 812 \\
\text { BM } 34221+\text { BM } 34299+\text { BM } 35119+\text { BM } 35206+\text { BM } 35445+\text { BM } 45702(=\text { Sp. } 327+\text { Sp. } 410+ \\
\text { Sp.II,664 }+ \text { Sp.II,763 }+ \text { Sp.II, } 1034+\text { SH 81-7-6,107 })
\end{gathered}
$$

Contents: Procedure text for Jupiter and Saturn
Arrangement: O/R
Provenance: Babylon [Sp. and SH]

## Section 1

Obv. I, beginning destroyed Transcription

Commentary
It is certain that this section concerns System B of the theory of Jupiter. This is shown by the occurrence of the following constants (cf. Introduction p. 311):

$$
\begin{array}{lrl}
\text { Lines } 2 \text { and } 4 & 6,31=\| & (=391 \text { mean synodic arcs/periods, added) }) \\
\text { line } 4 & 1 ; 48=d & \text { of } \Delta \mathrm{B} \text { or } \Delta \mathrm{T} \\
\text { line } 5 & 45 ; 14^{r}=u & \text { of } \Delta \mathrm{T} \\
\{\text { line } 6 & 50 ; 7,15^{r}=M & \text { of } \Delta \mathrm{T} \text { (added) }\} \\
\text { line } 6 & 40 ; 20,45^{\prime}=m & \text { of } \Delta T
\end{array}
$$


#### Abstract

The value $50 ; 7,15^{r}=M$ of $\Delta T$ is restored from the parallel passage of No. 813 Rev. I,5. The combination of these two sections shows that it was assumed that $m$ corresponds to $\mathbb{~} 15, M$ to )( 15 and consequently $u$ to $\theta 15$ and [15. In System A the apsidal line goes through mp 12;30 and )(12;30. Completely in the dark, however, remains the number ...]13,30,27,46 in lines 2 and 3 . In line 3 it seems to be called "mean value". The same number occurs in the parallel passage of No. 813 Section 12 but once broken into [13,30] at the end of of a line and 27,46 at the beginning of the next line. This may be either a real separation or simply split writing; no other passage shows a separation between 13,30 and 27,46. In No. 813 the number [13.30] 27,46 seems to be added to 45,14 which is the mean value of $\Delta T$."

In line 4 the difference $d=1 ; 48$ is multiplied by the number period $I I=6,31$. The result $(11,43 ; 48)$ is not preserved but there is hardly space left for more than the number which would represent the total variation $2(M-m) Z$ of $\Delta \mathrm{T}$. In line 2 the "mean value" (?) . . ] 13,30,27,46 is multiplied by II, again for unknown reasons."


Perhaps the most striking part of Neugebauer's analyses of ACT 812 Sections 1 and 2 is a puzzling inability to come to terms with what must surely be the elementary determination of the mean synodic period for Jupiter in mean synodic months followed by the same for the varying periods of System B. Plus, even though Section 1 is damaged, there is still sufficient information to understand precisely what the remaining parameters mean with confirmation found in the "alternate method" described in Section 2, which according to Neugebauer's translation teaches: ${ }^{59}$
$\ldots$.. the computation of the difference between synodic time and synodic arc (disregarding the trivial
12 months) following exactly formula (14), Introduction p. 286, for $i=1$.
[ Alternate (method): 33;8,]45, the mean(-value) of the longitudes, multiply by 0;1,50,40, and (you obtain) [1;8,8,]20. Add to it $11 ; 4$, and (you obtain) 12;5, $8,8,20$. Put it down for the gabari's of (one) year.
The first number, $33 ; 8,45^{\prime}$, is the mean synodic arc $\Delta \lambda=u$ in Systems B and $B^{\prime}$. The epact is $\epsilon=11 ; 4$ and consequently

$$
\frac{\epsilon}{6,0}=0 ; 1,50,40^{\prime} .
$$

Thus our text follows the formula $u \frac{\epsilon}{6,0}+\epsilon=12 ; 5,8,820$ '
The abbreviated value $12 ; 5,8$ ' is known from System $\mathrm{B}^{\prime}$ (cf. Introduction p. 311). A still more rounded-off value $12 ; 5,10^{\prime}$, is applied in the majority of the ephemerides. For the term gaba-ri cf. below, p. 413.

What follows must mean that the value $12 ; 5,8,8,20$ should be added to the synodic arc. Unfortunately, some of the corresponding signs are damaged, but one may perhaps read
[From (one) appear]ance to (the next) appearance, (the arc) between them you put down.
[12;5,8,8,20 you add to it and predict the [dates]s.

## X. Jupiter

Now had Neugebauer carried out the instructions he would have obtained $12 ; 5,8,8,20+33 ; 8,45=45 ; 13,53,8,20$ as opposed to the mean value " $45 ; 14^{\prime}=u$ of $\Delta T^{\prime \prime}$ " with constants for the maximum and minimum times also included between lines 5 and 6 of Section 1 . So just where was Neugebauer's problem? Apart from temporal racism, in the present case it can be traced to one line in the latter's translation and commentary for Section 2 of ACT 812 where he disregards "the trivial 12 months," except that elsewhere he stipulates that the 12 months should be added to to obtain the synodic times, which is indeed the last requirement to actually predict the dates. But one other earlier step, which is to convert back to days after carrying out key parts of the computation in "tithis" was not apparently carried out. As a result, his final representations of the synodic times for Jupiter using the above values remained in in mixed format. Thus 12 mean synodic months of $29 ; 31,50,8,20$ days to be added in the final step to values were instead combined by Neugebauer with results which were still in tithis yet still needed to be divided by 30 before the last addition of the 12 months. Therefore division of the end product in the text by 30, i.e., $45 ; 13,53,8,20 / 30$ $=1 ; 30,27,46,16,40$ plus the final addition of 12 mean synodic months results in a mean synodic period of Jupiter of $13 ; 30,27,46,16,40$ months (MSM). Which can then be compared to the three-place number 13;30,27,46 (MSM) in Section 1 - the unknown parameter which left Neugebauer "completely in the dark."
Once aware of the latter result, however, the application of the concept to the contents of lines 5 and 6 of Section 1 naturally follow, i.e., subtraction of the mean synodic arc $33 ; 8,45^{\circ}$ (located at Sagittarius 15 ) from $45 ; 14$, the like subtraction of the maximum synodic arc $M=38 ; 2^{\circ}$ from $50 ; 7,15$ (located at Pisces 15 ) and a final subtraction of the minimum arc $m=28 ; 15,30$ from $40 ; 20,45$ located at Virgo 15 all result in the more convenient constant 12;5,15.

It turns out that there are four constants: the most accurate being that derived in Section 2 of ACT 812, the next $12 ; 5,8$, followed by $12 ; 5,10$ and lastly, the above value $12 ; 5,15$ as the least accurate but the most convenient to use. In actual fact, the differences for the final periods in days for all four constants are quite small, e.g., using the mean synodic arc ( $u$ ) for Jupiter of $33 ; 8,45^{\circ}$ and MSM ( $29 ; 31,50,8,20$ days), the periods in MSM and days are:

| Jupiter k | $\mathrm{k}+33 ; 8,45$ | Divided by 30 | +12 MSM | Synodic S (days) |
| :--- | :--- | :--- | :--- | :--- |
| $12 ; 5,8$ | $45 ; 13,53$ | $1 ; 30,27,46$ | $13 ; 30,27,46$ | 398.890789 |
| $12 ; 5,8,8,20$ | $4 ; 13,53,8,20$ | $1 ; 30,27,46,16.40$ | $13 ; 30,27,46,16,40$ | 398.890827 |
| $12 ; 5,10$ | $45 ; 13,55$ | $1 ; 30,27,50$ | $13 ; 30,27,50$ | 398.891336 |
| $12 ; 5,15$ | $45 ; 14$ | $1 ; 30,28$ | $13 ; 30,28$ | 398.892703 |

Table 6. Babylonian Jupiter time constants and resulting periods ( $S$ ) in days.
with the slightly low constant 12;5,8 now explained by its use to generate the "mystery" period 13;30,27,46 MSM.
More critically, however, Neugebauer does not explain that the multiplication factors in Section 2 originate from the relation which incorporates "tithis" (thirtieths of the mean synodic month, ${ }^{\text {r }}$, i.e., $30 \cdot 12 ; 22,8 \mathrm{MSM}=371 ; 4$, with division by $360^{\circ}$ providing a time per degree for the motion of either Earth (or Sun) of $1 ; 1,50,40$. Which is where the simple addition of the mean synodic arc (u) $33 ; 8,45^{\circ}$ then $(u)+\mathrm{N} \cdot \mathrm{d}$ can be seen as a separate "multiplication" factor of 1 x from $1 ; 1,50,40$ with the results recorded in a separate column as the first half of the task with the second the corresponding times from the standard constant(s) based on $0 ; 1,50,40$ or $0 ; 1,50$, etc.

## XI. Determination of the Babylonian extremal synodic arcs/times and the varying values of System B

Up to a point this problem can be said to have already been dealt with by Neugebauer's detailed descriptions in THE PLANETARY THEORY IN GENERAL in Astronomical Cuneiform Texts (1955:284-287). Except for two fundamental issues, which is that no instructions were given by Neugebauer to generate a System B from scratch, despite his familiarity with the components necessary for this purpose. This is also why Aaboe (1964:32) was unable to develop a System B from the BM 37089 data for the Earth/Sun based on the difference $d=0 ; 0,1,32,42,13,20^{\circ}$ per day, and although incomplete, the equally extensive daily velocities in the text. The second deep-seated problem is how any progress could be expected in the absence of a fictive approach to planetary motion and/or some form of model.

Returning to this central issue in the context of ACT 812 Section 1 the method for establishing the apsidal synodic arcs becomes apparent from the residual parts of this text, specifically lines 2 and 4 with Neugebauer's comments in part also emphasizing the likely generation of the mean synodic time: ${ }^{39}$

[^2]Recalling again that the Babylonian long integer period relation for Jupiter is:

$$
\begin{equation*}
36 \text { revolutions (Z) and } 391 \text { synodic periods or arcs (II) in } 427 \text { (N) years } \tag{4}
\end{equation*}
$$

and now aware that the mystery number " $13,30,27,46$ " is unequivocally the mean synodic period for Jupiter in mean synodic months it is clear that the multiplication of this constant by 391 (the number of synodic periods for Jupiter) yields the total synodic motion over 427 years and 36 revolutions. Whereas the further multiplication of 391 by the difference $d=1.8(391 \bullet 1 ; 48)$ yields a total of 11,$43 ; 48(703.8)$. The division of the latter by 36 therefore provides the number of increments per synodic arc for one revolution $=19 ; 33$ (19.55) with the further division by two giving the amplitude $=9 ; 46,30$ (9.775). As in fact included by Neugebauer in his analysis of Babylonian System B for Jupiter where the subtraction of the minimum synodic arc $(m)$ of $28 ; 15,30^{\circ}$ from the maximum $(M)$ of $38 ; 2^{\circ}$ results in 9;46,30.

But this range - aphelion to perihelion - can be halved again to take advantage of the relation for the mean synodic $\operatorname{arc}(u)=1 / 2(m+M)$ whereas the parameter $P$ is $(391 / 36)$ and the full amplitude also $P d=19.55$. Therefore, with the inclusion of the mean synodic arc $(u)$ the maximum $(M)$ and minimum $(m)$ synodic arcs for System B can be derived from the general relation:

$$
\begin{equation*}
\text { Babylonian System B extremal synodic arcs }(M, m)=u \pm 1 / 4 \mathrm{Pd} \tag{6}
\end{equation*}
$$

From this viewpoint the period $T$ is dependent on the mean synodic arc $(u)$ with the only variable available $d$, the amount which increases from aphelion to perihelion and vice versa from the latter back to aphelion, etc.

## XII. System B locations and times for the extremal/apsidal synodic arcs of Jupiter

Based on the integer period relation (4) above, the parameters of System B for Jupiter reduced to their simplest form are:

11;51,40 (11.86111*) years for the period of revolution $T(427 / 36)$ 10;51,40 (10.86111*) years for the parameter $P(391 / 36,360 / u)$ and/or $T-1)$ $1 ; 48^{\circ}$ for the difference per synodic arc $d$ (variable(?), according to choice)
while relation (6) provides the extremal velocities $(m)$ and $(M)$ which, along with the mean synodic arc ( $u$ ) below completes the set:
$28 ; 15,30^{\circ}$ for the minimum synodic arc $(m)$
$33 ; 8.45^{\circ}$ for the mean synodic arc $(u)$
$38 ; 2^{\circ}$ for the maximum synodic arc $(M)$

Corresponding times - as shown in Table 14 - are dependent on the choice of constant (i.e.,12; 5,8 through 12;5,15). However, System B' used in ephemeris ACT No. 640 and procedure text ACT No. 813 Sections 21 and $22^{60}$ utilize a value for $d$ of $1 ; 46,40$ instead of the standard value $1 ; 48$ which, with $u$ and $P$ remaining constant, results in new values for $M$ and $m$ based on $1 / 4 P d=4 ; 49,37,46,40$ with extremal velocities of $28 ; 19,7,13,20^{\circ}$ and $37 ; 58,22,46,40^{\circ}$. All three may be compared to the equivalents given by Neugebauer of $9 ; 39,10^{\circ}$ (the half $=4 ; 49,35$ is not included by the latter) followed by $28 ; 19,10^{\circ}$ and $37 ; 58,20^{\circ}$ deduced from the above texts. Neugebauer's treatment of this variant includes both a change in the underlying period while still maintaining the standard mean synodic arc of $33 ; 8,45^{\circ}$. It also contains "certain discrepancies . . .which are not exclusively due to rounding-off for the sake of easier computation." These include the observation that replacement of the constant 12;5,8,8,20 for the mean synodic
time with the lesser value 12;5,8 is "a permissible approximation." And certainly, rounding applied to the four-place data to reduce it to two-places is indeed reasonable. Yet there remains a problem since Neugebauer's constants for the positions and times also vary. This is no small matter, although it does serve to emphasize that the methodology, - despite numerous variants and details - was not fully understood, especially when tithis and the mean synodic month are involved, as in the present case.

More to the point, Neugebauer also provides the following commentary concerning a "third method" supplied in Sections 14,15 and 16 of ACT 813 for System A' oddly exemplified by the use of much rounded parameters for the velocity of the "Sun." Thus he writes:

> In spite of many difficulties in the reading of these sections, which are headed ' "third (method)", the main contents can be established. We are dealing with the slow, medium, and fast arcs of System A'. The synodic arcs 30 and and $33 ; 45$ are given, and the corresponding synodic times are to be found. In order to travel one degree the mean sun requires $\frac{1 ; 0,57,13}{0 ; 59,8}=1,50,49, \ldots$. If we abbreviate this value we obtain for a synodic arc of $w^{\circ}$ a time of $w+0 ; 1,50 w$. We know furthermore that 12 lunar months are $11 ; 3,20$ ' shorter than one solar year (lines $10,11,15$ ). Thus the time difference in question is found in Section 15 as follows: (added: zodiacal names included for the signs)

From 9 Cancer (sic! for Virgo) to 2 Capricorn (from) year to year $33 ; 45 \ldots$ (from) year to year 33;4[5 ....] multiply by $0 ; 1,50$ and (the result) $1 ; 1,52,30$ add to $33 ; 45$. $34 ; 46,52,[30]$ and $11 ; 3,20$, the days of the sun, add together and 45;50,[12.30 put down for the gaba-ri].

In Section 16 the factor 0;1,50 appears in the form 0;2-0;0,10:
From 2 Capricorn to 17 Taurus (from) year to year $36 \ldots .[\ldots] \ldots$. . ......... ]..... $36 \ldots$.
multiply by $0 ; 2$ and [(the result) $1 ; 12 \ldots \ldots \ldots \ldots . .36$ multiply by $0 ; 0,10$ and] subtract $0 ; 6$
from 1;12 and what remains (namely) 1;6 [add to 36 and to $11 ; 3,20$. The result is $48 ; 9,20$

In Section 14 one should expect for the slow arc a time difference of $30 \cdot 1 ; 1,50=30 ; 55$ whereas the text gives only 30;45:

From 9 Cancer to 9 Scorpio ] 30 (degrees or 1 ) danna and (from) year to year 11;3,20 days sub-
tracted(?) and (?) 30 (degrees or 1) dan[na . ... . .] . . . of Jupiter (from) year to year ...
$30 ; 45$ days multiply(?) and $30 ; 45$ days [and $11 ; 3,20$ ] days add together and $41 ; 28,20$ days
put down for the gaba-ri of (one) year.
Before discussing the above and the accompanying implications, there is, however a more immediate issue which concerns Neugebauer's introduction above of the ratio " $1 ; 0,57,13 / 0 ; 59,8$ " and an approximate result of " $1,50,39 \ldots$... This is a mystery, since in Section 13 just before (which is parallel to Section 2), his translation included 0;1,50,40 en route to the determination of the constants for $33 ; 8,45$ (Jupiter (u), i.e., intermediate values to be added ( $1 ; 1,8,8,20$ and $11 ; 4$ ) to obtain the final value $12 ; 5,8,8,20$. This is followed (as in the parallel text in Section 2) with condensed instructions for its use, which are worth repeating:

Alternate (method): $33 ; 8,45$, the mean(value) of the longitudes, multiply by $0 ; 1,50,40$, and (you obtain) $12 ; 8,8,20]$. Add to it $11 ; 4$, and (you obtain) $12 ; 5,8,8,20$. Put it down for the gaba-ri of (one) year. The arc [between them you put down. 12;5,8,8,20 add to it and predict the dates. 1,1,8,820 . .]

The reason behind labouring this point is that there seems to have been insufficient understanding of the central role played by the inter-related trio of luni-solar constants applied in Babylonian astronomy, namely, with assigned abbreviations introduced earlier, the following:

$$
\begin{aligned}
& \text { 1. Mean Synodic Month }(\text { MSM })=29 ; 31,50,8,20 \text { days }(29.530594135802474) \\
& \text { 2. Sidereal Year in MSM (MYR })=12 ; 22,8 \text { Mean Synodic Months }\left(12.36888^{*}\right) \\
& \text { 3. Sidereal Year in Days (SYR })=\text { SYR }=365 ; 15,38,17 ; 44,26,40 \text { days }(365.260637688614546)
\end{aligned}
$$

to which can be added the Zodiacal "Year" of $360^{\circ}$ with its $12 \cdot 30^{\circ}$ months of the Zodiac and the associated relations:

$$
\begin{aligned}
& \text { MSM•MYR }=\text { SYR } \quad(\text { MYR }=12 ; 22,8 \text { MSM }) \\
& 30 \bullet \text { MYR }=371 ; 4 \quad\left(6,11 ; 4=371.10666^{*} \text { tithi }\right)
\end{aligned}
$$

Thus arriving at the multiplication factor for the conversion to tithi per degree of 1;1,50,40 as used in the texts, along with a neat and simple method for concurrently recording the varying synodic arcs and their corresponding times.

$$
\frac{30 \cdot M Y R}{360}=\frac{\text { MYR }}{12}=\frac{6,11 ; 4}{6,0}=1 ; 1,50.40\left(1.03074074074^{*}\right)
$$

Which is to bifurcate the multiplication factor $1 ; 1,50.40$, i.e, first multiply the synodic arcs by the integer 1 , then set this aside (or record) the "result" as the arc under consideration for subsequent positional assignment(s) in degrees, remembering that the conversion to tithis will later require conversion to days. Meanwhile, with unity (1) removed the remainder of the multiplication factor $(0 ; 1,50,40)$ can be applied to the arc itself. Thus essentially an operation in five stages:
(1) Synodic arc• $1 ; 1,50,40$ for the arc with or without the difference (s) $d$.
(2) Syndic arc $0 ; 1,50,40+\mathrm{k}=11 ; 4$ (or 11;3,20)
(3) Step (1) combined with Step 2.
(4) (Step (3) sum divided by 30 to convert to MSM
(5) 12 MSM added to Step 4 for the corresponding Time in MSM
with, for the second part, the simpler two-place constant 0;1,50 as used in ACT 813, Sections 14 to 16.
Except that instead of MYR $=12 ; 22,8$ MSM the length of the year based on the replacement of $11 ; 4$ by 11;3,20 is $12 ; 22,6,40(12.36851851851852 \mathrm{MSM})$, the total in tithis is $371 ; 3,20\left(371.0555^{*}\right)$ and the multiplication factor now $1 ; 1,50,33,20$. Which further explains why the later was reduced to $01 ; 1,50$ while also retaining the 1 X factor for the for the synodic arc.

## XIII. Sections 14, 15 and 16 of ACT 813 revisited

The major point of interest is that the $30^{\circ}, 33 ; 75^{\circ}$ and $36^{\circ}$ synodic arcs of System $\mathrm{A}^{\prime}$ examined earlier with respect to the Fibonacci ratios found in the variants of relations 7,8 and 12 are here assigned specified sectors in the Zodiac:
$30^{\circ}$ Minimum arc $(m)$ assignment: 9 Cancer $\circlearrowleft$ to 9 Scorpio $m$.
$33 ; 45^{\circ}$ intermediate arc $\left(u_{2}\right)$ assignment: 9 Scorpio $m$ to 2 Capricorn bs
$36^{\circ}$ Maximum arc $(M)$ assignment: 2 Capricorn $\wp$ to 17 Taurus $૪$
with a line of apsides from 12;30 Virgo mp to 12;30 Pisces 00 according to Neugebauer and Act 814, Section 2.61


Fig 8. Jupiter System $A^{\prime}$ assigned sectors for the $30^{\circ}, 33 ; 45^{\circ}$ and $36^{\circ}$ synodic arcs
At which point it seems necessary to investigate this matter further in terms of elliptical orbits in general in light of the above assignments and previous discussions concerning the Babylonian synodic arcs.

Before returning to the ellipse in Babylonian contexts mentioned earlier, it may be helpful for the casual reader to revisit Kepler's three laws of planetary motion described and qualified by Cesare Emiliani (1995) as follows: ${ }^{62}$
"FIRST LAW: ' The orbit of a planet is an ellipse with the Sun at one of the two foci.' This law establishes the shape of planetary orbits. They are not circles, but ellipses.

SECOND LAW: ' A planet revolves around the Sun with the connecting lines sweeping equal areas in equal times.'
This law establishes that the planets move faster when they are closer to the Sun and more slowly when they are farther away.

THIRD LAW: 'The square of the sidereal period of a planet is proportional to the cube of the semimajor axis of it's orbit.'
This law establishes that the sidereal period of the planets (the time it takes to go around the Sun once) increases as the distance from the Sun increases."

Cesare Emiliani, THE SCIENTIFIC COMPANION, (1995:121)
For ellipses in the present context modern textbooks generally provide the mean distances and eccentricities in a priori formulas for the calculation of apsidal data, but what about the earlier Babylonian methodology ? As it so happens an alternative approach to the subject is provided in Robin S. Green's description of "Kepler's equation for a bound orbit," published in Spherical Astronomy (1985), the relevant parts of which are as follows: ${ }^{63}$
"... $F(r)$ is simply shorthand for the quadratic form given by

$$
\begin{equation*}
F(r)=r^{2}-2 a r+h^{2} a / u . \tag{6:19}
\end{equation*}
$$

Equation (6:18) implies that $F(r)$ is $\leq 0$. So we may conclude that the radius vector $r$ is restricted to the range

$$
\begin{equation*}
r_{1} \leq r \leq r_{2} . \tag{6:20}
\end{equation*}
$$

where $r_{1}$ and $r_{2}$ are the two roots of $F(r)=0$. In fact, they must correspond to the perihelion and aphelion distances. It should be noted, from equation (6:19), that $r_{1}+r_{2}=2 \mathrm{a}$. Moreover, it is convenient to introduce the parameter
$e$ - later to be identified with the orbital eccentricity - as

$$
\begin{equation*}
e=\frac{r_{2}-r_{1}}{r_{2}+r_{1}} \tag{6:21}
\end{equation*}
$$

Then, working in terms of $a$ and $e$, it is seen that $r_{1}=a(1-e), r_{2}=a(1+e)$

Thus in the present context, all that will be required is the conversion of the Babylonian extremal/apsidal synodic arcs from degrees to time ( $T$ ), then (via the harmonic law) to distance ( $R$ ) to conform with relations (6:20) and (6:21) for the eccentricity and the remaining parameters.

Finally, in terms of background information it may prove beneficial to re-evaluate Green's relation (6:21) applied to the well-known ellipse based on the 5-4-3 Pythagorean triple in both standard and astronomical contexts:


Fig 9. I, II. The ellipse based on the 5-4-3 Pythagorean set in both general and astronomical contexts
Here the integral relationship between the semi-major axis 5 ( $a$ ), the semi-minor axis 4 (b) and 3, the parameter (c) is routinely expressed as $a^{2}=b^{2}+c^{2}$ with the major axis $2 a=10$, the minor axis $2 b=8$, the distance between the two foci $=2 c$ and eccentricity $(e)$ from $c / a=0.6$.

## XIV. Apsidal distances for the elliptical orbit of Jupiter

System A maximum synodic arc $(W)$ of $36^{\circ}$ and minimum synodic arc ( $w$ ) of $30^{\circ}$ of Jupiter can be applied to the following variant of relation (13): $\quad P=360^{\circ} / u=T-1, T=P+1$
i.e., from $T=(P+1)$ and :- Heliocentric distances $R=\left[\left(360^{\circ} / \text { Synodic Arcs } W, w\right)+1\right]^{2 / 3}$
to generate (via the harmonic law or variant) the following apsidal data for Jupiter with reference to unity ("a.u").

> Perihelion distance $R$ from $\left(360^{\circ} / 36^{\circ}\right)+1, T=11, R=11^{2 / 3}=4.9460874$ (a.u.)
> Aphelion distance $R$ from $\left(360^{\circ} / 30^{\circ}\right)+1, T=13, R=13^{2 / 3}=5.2374311$ (a.u.)
> $(e)=(A p h R$ - Perh $R) /(\operatorname{Aph} R+$ Perh $R)=0.05562721$ (modern: 0.04849485$)$
> (determination of eccentricity e, R. M. Green, Spherical Astronomy 1985:141)

The same process applied to System B extremal synodic arcs for both Jupiter and Saturn results in lower correlation for the eccentricities, whereas System A extremal synodic arcs for Saturn prove to be the most useful.

## XV. Apsidal distances for the elliptical orbit of Saturn

Saturn's System A apsidal synodic arcs generate corresponding heliocentric distances from the maximum (W) synodic arc $14 ; 3,45^{\circ}$ and ( $w$ ), the minimum synodic arc $11 ; 43,7,30^{\circ}$ as follows, with rounded distances included for a more immediate Babylonian application:

$$
\begin{aligned}
& \text { Perihelion } R \text { from ( } 360 / W \text { ) }+1, T=26.60, R=T^{2 / 3}=8.9108902 \text { (a.u.), rounded: " } 9 " \\
& \text { Aphelion } R \text { from ( } 360 / w \text { ) }+1, T=31.72, R=T^{2 / 3}=10.0204861 \text { (a.u.), rounded: " } 10 \text { " } \\
& \text { Mean Distance (265/9, Mean period, } T, \quad R=T^{2 / 3}=9.5353267 \text { (a.u.) rounded: " } 9.5 \text { " } \\
& \text { Distance between foci }(d)=1.109595829, c=0.5547979145 \text {, rounded: " } 1 \text { " \& " } 1 / 2 \text { " } \\
& \text { ( } e=0.0586115 \text {; rounded } e=1 / 19=0.0526316 \text {. modern } e=0.0555086 \text { ) }
\end{aligned}
$$

All of which come into finer focus when applied to a Babylonian mathematical cuneiform text of hitherto unknown practical significance. As described by Marcus du Sautoy, ${ }^{64}$ the text involves a procedure known as "completing the square," which - as du Sautoy notes - is equivalent to solving the quadratic equation $x^{2}-x=870$.

The latter's description of the text is given next in decimals for clarity with apsidal distances for Uranus similarly assigned to the right of the original text and the rounded data for Saturn applied next as the working example.
I have subtracted the side of my square from the area: 870. I have subtracted the side of my square from the area: 380 .

You write down 1, the coefficient.
You must break off half of 1 .
0.5 and 0.5 you multiply. You add 0.25 to 870 . Result 870.25.

This is the square of side 29.5.
You add 0.5, which you multiplied, to 29.5.
Result 30, the side of the square. [ ORIGINAL text ]
[Quadratic equation : $\mathrm{x}^{2}-\mathrm{x}=870$ ]


You write down 1, the coefficient.
You must break off half of 1.
0.5 and 0.5 you multiply. You add 0.25 to 380 . Result 380.25 .

This is the square of side 19.5.
You add 0.5, which you multiplied, to 19.5 .
Result 20, the side of the square. [ URANUS (added) ]
[ Quadratic equation: $x^{2}-x=380$ ]
The original Babylonian Mathematical Problem
I have subtracted the side of my square from the area: 870. You write down 1, the coefficient. You must break off half of 1 .
0.5 and 0.5 you multiply.

You add 0.25 to 870 . Result: 870.25
This is the square of side 29.5.
You add 0.5 , which you multiplied, to 29.5
Result 30, the side of the square."
Method applied to the heliocentric elliptical orbit of
Saturn based on rounded Babylonian System A data.
(Added : Quadratic equation: $x^{2}-x=90$; eccentricity $e=1 / 19$ )
I have subtracted the side of my square from the area: $90=b^{2}$
You write down 1, the coefficient. $\quad d^{*}=1$
You must break off half of $1 . \quad \frac{1}{2} d=c=0.5$
0.5 and 0.5 you multiply. $\quad c^{2}=0.25$

You add 0.25 to 90 . Result: $90.25 \quad c^{2}+b^{2}=90.25$
This is the square of side 9.5. From $a=\sqrt{ }\left(b^{2}+c^{2}\right)$
You add 0.5 , which you multiplied, to $9.5(c+a)=10$, aphelion $r$
Result 10, the side of the square. Saturn, aphelion $r$
also, (added) Subtract 0.5 from $9.5=9 \quad$ Saturn, perihelion $r$
Saturn, mean distance : $\boldsymbol{R}=\mathbf{9 . 5}$ (relative to unity)
${ }^{*} d=$ difference between aphelion and perihelion distances

Fig 10. The Babylonian mathematical problem and elliptical orbit of Saturn from rounded Babylonian System A data.

Similarly applied, with fixed "coefficients," $d=1, c=0.5$, the data for Uranus and Neptune are therefore.

Uranus: $\left(b^{2}=380\right)$, add 0.5 to $19.5(c+a)=20$, aphelion $r$
(Added: Quadratic equation: $x^{2}-x=380$; eccentricity $e=1 / 39$ )
(Added), subtract 0.5 from $19.5(a-c)=19$, perihelion $r$ Uranus, mean distance : $\boldsymbol{R}=\mathbf{1 9 . 5}$ (relative to unity)

Neptune: $\left(b^{2}=870\right)$, add 0.5 to $29.5(c+a)=30$, aphelion $r$ (Quadratic equation: $x^{2}-x=870$, the original problem ; $e=1 / 59$ ) (Added), subtract 0.5 from $29.5(a-c)=29$, Perihelion $r$ Neptune, mean distance: $\boldsymbol{R}=\mathbf{2 9 . 5}$ (relative to unity)

## XVI. Uranus and Neptune

The two theoretical positions exterior to Saturn require that the "coefficient" $d$ (1) and its half $c$ remain constant, thus the eccentricities diminish with distance commencing with "sides of squares" of 10 (as used), 20 (unattested, but sequential), and 30, again sequential and the original target. In other words, the aphelion distances of 20 for Uranus ( $e=1 / 29$ ) and 30 a.u. for Neptune ( $e=1 / 59$ ). Lastly, on a more practical note, unaided detection of Uranus cannot be ruled out in any case; the latter, although faint is undoubtedly visible to the naked eye. ${ }^{65-67}$ Alternatively, if the 2-(3)-5 Fibonacci series is theoretically assigned to the periods of Saturn, Jupiter-Saturn synodic SD1 and Jupiter, then the series will continue outward for two more planetary positions before ultimately ending at unity.

Thus commencing with the period of Neptune (1), Uranus-Neptune synodic also1 and the period of Uranus 2 the sequence is followed next by 1-(2)-3 to include the latter. Continuing inwards with the resonant Saturn-Jupiter triple [ 2-(3)-5 ] the sequence then proceeds in due order with 3-(5)-8, 5-(8)-13, 8-(13)-21, 13-(21)-34 and 21-(34)-55, etc. Thus essentially the inclusion of the intermediate synodic cycles to complete the Pierce-Agassiz model which finally became the extended Phi-series planetary framework.

Moreover, though wandering far afield, the above requirement that the "coefficient" d remains unity (1) suggests that certain "Pythagorean" matters involving the number 216 in Vitruvius' Ten Books of Architecture might have some bearing on the discussion. In particular, what may best be described as a "columnar" analogy, where the following three architectural "styles" are grouped together, as are the mean distances of 9.5, 19.5 and 29.5 for the Saturn, Uranus and Neptune ellipses assigned above, with the "coefficient" ( $d=1$ ) and half ( $c=1 / 2$ ) incorporated thus: ${ }^{68}$

In the systyle, let the height be divided into nine and a half parts, and one of these given to the thickness of the column.. If the building is to be systyle and monotriglyphic, let the front of the temple if tetrastyle, be divided into nineteen and a half parts; if hexastyle ${ }_{\llcorner }$into twenty-nine and a half parts. (emphases supplied)

VITRUVIUS: The Ten Books on Architecture, Bk III, Ch. III,10. (trans. M H. Morgan,1960:84)

## XVII. Completing the Square

The application of rounded System A parameters for Saturn permits a comparable solution to the problem and also supplies a practical meaning which can be expanded to include theoretical distances for both Neptune and Uranus. This does not, of course, establish that either of the orbits of the outermost planets were known per se, but rather, that the suggested data can be interpreted as theoretical extensions beyond Saturn, the outermost planet known in Antiquity. As for "completing the square, this is readily demonstrated by dual figures for the Saturn data:

\begin{tabular}{|c|c|}
\hline \begin{tabular}{l}
"Completing the square" text applied to Saturn using System A distances relative to unity (or "a.u."): \\
Text applied to Saturn using the following parameters: \\
Perihelion \(R=9\), mean \(R=9.5\) and aphelion \(R=10\). \\
I have subtracted the side of my square from the area: 90 . You write down 1, the coefficient. \\
You must break off half of 1 . \\
0.5 and 0.5 you multiply. You add 0.25 to 90 . Result 90.25. \\
This is the square of side 9.5 . \\
You add 0.5 , which you multiplied, to 9.5 . \\
Result 10 , the side of the square. \\
[ Original Area \(=10^{2}=100 ;\) not given ] \\
Value of the "Side" \(=1 \times 10=10\) \\
Subtraction of "side" from the area \(=90\) \\
Addition of \(\mathrm{c}^{2}(0.25)=90.25\) \\
Square root of \(\left(b^{2}+c^{2}\right)=9.5=\) mean \(R\) \\
Addition of \(\mathrm{c}=0.5\) to mean \(R=10\), the \\
Side of the square and aphelion \(R\). \\
Available: Perihelion \(R=9\) (mean \(R-c(0.5)\). \\
( \(10 \times 10\) Square not to Scale )
\end{tabular} \& 1

$1 / 10$ <br>
\hline
\end{tabular}

10
Fig. 11b. Completing the square procedure applied to Saturn.

Theoretical elliptical orbit for "Saturn"
Fig. 11a. Saturn ellipse with rounded System A distances.

Thus aphelion distances of 20 (a.u.) for Uranus and 30 (a.u.) for Neptune with the latter's mean distance of 29.5 a.u providing a B1 base period of 160.2260124 years and 164.3167673 years from the solution to the problem (" 30 "). All of which is sensibly heliocentric in both form and application with the mean heliocentric distances known and aphelion distances the target. Which also makes perfect sense, since the mean and aphelion distances are all that are required to supply the remaining parameters for practical ellipses with assigned lines of apsides. Therefore, if indeed astronomical in the above sense, then prior tasks from a modern viewpoint would entail the calculation of cube roots from the squares of the periods of revolution ( $T$ ), thus "Kepler's" two-parameter law of planetary motion: $R=T^{2 / 3}$. As it turns out, the method used was identical in so much as the final product was still $R=T^{2 / 3}$ but includes the corresponding velocity with respect to unity $(V r)$ and inverse $\left(V_{i}\right)$ exemplified by the Triple interval $\left[3^{0}, 3^{1}, 3^{2}, 3^{3}\right.$ ] $=[1,3,9,27]$. But before continuing in this vein the introduction of orbital velocities requires further expansion.

## XVIII. Galileo, Plato, and orbital velocity in earlier times

Although including the time of Plato and latter's Dialogues some 400 years before the current era there is another matter which needs to be considered, which is orbital velocity in earlier times. Specifically, velocity expansions for the laws of planetary motion published in the Journal of the Royal Astronomical Society of Canada in a 1989 paper by the present scriber ("Projectiles, Parabolas and Velocity Expansions of the Laws of planetary motion") based on Galileo's treatment of orbital velocity in his Dialogues Concerning Two New Sciences (1638)..9 Predicated on Plato's Timaeus and Epinomis readers are invited to confirm Galileo's reasoning, when, after explaining his understandable hesitancy concerning open discussion of the subject, he states: ${ }^{70}$
"but if anyone desires such information he can obtain it for himself from the theory set forth in the present treatment."
Which is how the present scriber - little more than a tertiary restorer - came to be involved in this matter. But either way the final products were assuredly an improvement over the two-parameter period $(T)$ and distance $(R)$ format currently in use. In other words, instead of the Harmonic or Kepler's Third law of planetary motion: $R^{3}=T^{2}$, a series of additional formulas were now available incorporating orbital velocity $\operatorname{Vr}$ (relative to unity) and inverse Velocity (Vi). Thus the following velocity relations for the laws of planetary motion with related notes as published in 1989: ${ }^{68}$

$$
\begin{align*}
& " R=V_{\mathrm{i}}{ }^{2}  \tag{1}\\
& V_{\mathrm{r}}=R / T,  \tag{2}\\
& V_{\mathrm{i}}=T / R,  \tag{3}\\
& T=V_{i^{3}},  \tag{4}\\
& V_{\mathrm{i}}^{6}=R^{3}=T^{2}, \tag{5}
\end{align*}
$$

where $T$ is the sidereal period in years, follow from Kepler's Third Law of planetary motion. Relation (5) may also be expressed in exponents (i.e., $\left[V_{i}{ }^{0}, V_{i}{ }^{1}, V_{i}{ }^{2}, V_{i}{ }^{3}\right]$ ) and applied to the parabola as the first three integer sets which illustrate the Third Law.?

NOTES 5 and 7
5. Galileo discusses the sets $[1,2,4,8]$ and $[1,3,9,27]$ with respect to squaring and cubing in the New Sciences,(First Day [83]). The same sets are also mentioned by Plato in the Timaeus ( 35 b and 43 d ), and the first set $[1,2,4,8$ ] is discussed again in Epinomis (991a-992a). Added, 2022: [1,1,1,1] (Earth), [1,2,4,8] (no body), [1,3,9,27] (Saturn, perihelion Vi, R,T) 7. The ancient relationship between a point, a line, an area, and a volume. See Galileo's discussion of the latter pair and the "sesquialteral ratio" between them in the Two New Sciences, First Day, (134-135)."
The paper also included a comparison of the mean velocities of the planets utilizing additional ratios and also a constant velocity for Earth in kilometers per second to prove the value of the approach in practical terms.

| MEAN Planetary Distances, MEAN Periods and MEAN Velocities* |  |  |  |  |  |
| :--- | ---: | ---: | ---: | :---: | ---: |
| Planet | Distance $(R)$ | Period $(T)$ | $V_{r}(R / T)$ | $V_{a}(k R / T)$ | Modern |
| Mercury | 0.387107 | 0.24085 | 1.60725 | 47.88 | 47.89 |
| Venus | 0.723350 | 0.61521 | 1.17577 | 35.03 | 35.03 |
| Earth | 1 | 1 | 1 | 29.79 | 29.79 |
| Mars | 1.523733 | 1.88089 | 0.81011 | 24.13 | 24.13 |
| Jupiter | 5.201287 | 11.86223 | 0.43847 | 13.06 | 13.06 |
| Saturn | 9.538188 | 29.45770 | 0.32379 | 9.65 | 9.64 |
| Uranus | 19.182303 | 84.01390 | 0.22832 | 6.80 | 6.81 |
| Neptune | 30.057937 | 164.79300 | 0.18240 | 5.43 | 5.43 |
| Pluto | 39.440188 | 247.69000 | 0.15923 | 4.74 | 4.74 |

[^3]
## XV. The Double and Triple Intervals revisited;

The Double and Triple Intervals: [ $\mathbf{1}, 2,4,8$ ] and $[\mathbf{1}, 3,9,27]$ are the first pair of integers which demonstrate the the Harmonic Law relative to unity: $R^{3}=T^{2}\left(4\right.$ a.u. $.^{3}=8$ years ${ }^{2}=64$, and 9 a.u. ${ }^{3}=27$ years $\left.{ }^{2}=729\right)$. Plus, in the same sense, the "Single" interval [ 1, 1, 1, 1 ] ] assigned to Earth ( 1 a.u. ${ }^{3}=1$ year ${ }^{2}=1$ ) with the second parameter also providing the names for the intervals, but otherwise (in modern times) of unknown significance.

Surprisingly (and in whatever Babylonian context), the determination of distances relative to unity is nonetheless impressive in so much as it incorporates relative orbital velocity $(V r)$ and inverse ( $V$ Vi) to streamline the process and simplify calculation. This much can be surmised from the Babylonian texts under discussion, augmented by the Pythagorean Tetractys, Plato's Double and Triple Interval plus Old Babylonian mathematical text VAT 8457 for the eventual calculation of cube roots from the periods of revolution ( $T$ ).

Even so, there is one additional complication that arises from the simplicity of the final product which is that the corresponding heliocentric distance $(R)$ is not obtained from the cube root of the period squared ( $R={ }^{3} \checkmark T^{2}$ ) but the cube root of the period $(T)$ itself as the initial goal, e.g., the cube root of 27 years is 3 , which when squared yields the required distance $(R)=9$. What is the astronomical significance of the cube root of the period of revolution?

Simply stated and denoted by $\mathbf{V i}$, this value is the inverse orbital velocity with respect to unity corresponding to the square root of the heliocentric distance $\boldsymbol{R}$ and the cube root of the period of revolution $\boldsymbol{T}$. Which together, expressed in a format which extends beyond the two-parameter limitations of "Kepler's" Third Law ( $R^{3}=T^{2}$ ) is:

$$
\begin{aligned}
\text { Velocity } \boldsymbol{V i} \mathbf{i}^{6}= & \text { Distance } \boldsymbol{R}^{\mathbf{3}}=\text { Revolution } \boldsymbol{T}^{\mathbf{2}} \\
& \text { Distance } \boldsymbol{R}=\text { Velocity } \boldsymbol{V} \mathbf{i}^{\mathbf{2}}
\end{aligned}
$$

(relation 5, 1989)
(relation 1, 1989)
with possible origins, it is suggested, demonstrated in Old Babylonian mathematical text VAT 8547.

## XVI. VAT 8547 and the extraction of cube roots from periods of revolution

The present approach to the extraction of cube roots in VAT 8547 and similar texts differs from previous treatments which were handicapped by lack of context, whereas it is suggested here that they pertain to astronomy in general and the cube roots of the periods in particular. Which, as it so happens, are also the square roots of the distances, and therefore together they provide the basis (along with unity) for the "interval" concept, tetradic exponents, "Point Line-Square-Cube analogy" and "completing the cube" methodology. Even so, this does not explain the oddities in VAT 8547 discussed by Neugebauer and Sachs (1945), ${ }^{71}$ Sachs (1952) ${ }^{72}$ and Muroi (1989). ${ }^{73}$ Nonetheless, what follows next is gained from these various commentaries, all linguistic complexities notwithstanding. Matters are, however simplified in so much as the same procedure is applied to all four numbers (or years), i.e., 27, 64,125 and 216 with the procedure for the cube root of the first number (27) translated by Kazuo Muroi as follows: ${ }^{73}$

```
What is the cube root of 27?
(When) you (perform the calculation), subtract 0;7,30 from 27 and
you will leave 26;52,30. The 0;7,30 which you subtracted
place below 26;52,30 and
26;52,30 0;7,30. What is the cube root of 0;7,30? 0;30.
Make the reciprocal 0;7,30 and (y ou get) 8
Multiply }8\mathrm{ by 26;52,30<+>0;7,30 (and you get) 3,36.
What is the cube root of 3,36? 6.
Multiply }6\mathrm{ by 0;30, the cube root (and you get) }3\mathrm{ .
3 is the cube root of 27.
```

which, with decimals substituted for sexagesimal numbers and fractions added, is:

```
What is the cube root of 27?
(When) you (perform the calculation), subtract 0.125 (1/8) from 27 and
you will leave 26.875. The 0.125 (1/8) which you subtracted
place below 26.875 and
26.875 0.125 (1/8). What is the cube root of 0.125 (1/8)? 0.5 (1/2).
Make the reciprocal 0.125 (1/8) and (you get) }8
Multiply }8\mathrm{ by 26.875<+>0.125 (and you get) 216.
What is the cube root of 216? }6
Multiply 6 by 0.5 (1/2), the cube root (and you get) 3.
3 is the cube root of 27.
```

The instructions in lines 5 and 7 have caused the most difficulty; but the latter $(26.875 / 0.125=215)$ is followed by 216 , i.e., with 1 added to "complete the cube." The procedure uses four inter-related constants applied to 27, 64, 125 and 216 in VAT 8547 to obtain 3, 4, 5 and 6 for the cube roots with (+1) added in Step 3 as follows:

1. The fraction $(1 / 8)=0.125$ to be subtracted from the number whose cube root is to be determined.
2. The integer 8 as a constant multiplier.
3. Unity $(1)=8 \cdot(1 / 8)$ to be added (i.e., recombined) to "complete the cube" for which an integer cube root is known.
4. A fixed divisor $0.5(1 / 2)$ is applied to produce the final cube root of the period with the understanding that the cube root of $1 / 8$ is $1 / 2$.

Therefore, for the Ternary, Quaternary, Quinary and Senary "periods" in VAT 8547 of 27, 64, 125 and 216 (years), the cube roots are $3,4,5$ and 6 obtained in the following consistent manner which (for completeness and to connect with the procedure outlined above) includes the Single [ 1, 1, 1, 1 ] and Double interval [ 1, 2, 4, 8 ] given below:

SINGLE INTERVAL (UNITY): [ 1, 1, 1 a.u., 1 year ]

1. Subtraction of 0.125 from 1 year $(1-0.125)=0.875$.
2. Multiplication of 0.875 by $8=7$.
3. Understanding that $(7+1)=8$, a number with a known integer cube root (2)
4. Reduction of the known integer cube root 2 by $1 / 2$ for the cube root of $1=1$.

DOUBLE INTERVAL (BINARY): [ 1, 2, 4 a.u., 8 years ]

1. Subtraction of 0.125 from 8 years $(8-0.125)=7.875$.
2. Multiplication of 7.875 by $8=63$.
3. Understanding that $(63+1)=64$, a number with a known integer cube root (4)
4. Reduction of the known integer cube root 4 by $1 / 2$ for the cube root of $8=2$.
followed by the cube roots for the four numbers in VAT 8547 ( $27,64,125,216$ years ) :
TRIPLE (TERNARY) INTERVAL: [ 1, 3, 9 a.u., 27 years ]
5. Subtraction of $0.125(1 / 8)$ from 27 years $(27-0.125)=26.875$.
6. Multiplication of 26.875 by $8=215$.
7. Understanding that $(215+1)=216$, a number with a known integer cube root (6).
8. Reduction of the known integer cube root 6 by $1 / 2$ for the cube root of $27=3$.

QUADRUPLE (QUATERNARY) INTERVAL: [ 1, 4, 16 a.u., 64 years ]

1. Subtraction of $0.125(1 / 8)$ from 64 years $(64-0.125)=63.875$.
2. Multiplication of 63.875 by $8=511$.
3. Understanding that $(511+1)=512$, a number with a known integer cube root (8).
4. Reduction of the known integer cube root 8 by $1 / 2$ for the cube root of $64=4$.

QUINTUPLE (QUINARY) INTERVAL: [ 1, 5, 25 a.u., 125 years ]

1. Subtraction of $0.125(1 / 8)$ from 125 years $(125-0.125)=124.875$.
2. Multiplication of 124.875 by $8=999$.
3. Understanding that $(999+1)=1000$, a number with a known integer cube root (10).
4. Reduction of the known integer cube root 10 by $1 / 2$ for the cube root of $125=5$.

SEXTUPLE (SENARY) INTERVAL: [ 1, 6, 36 a.u., 216 years ]

1. Subtraction of 0.125 from 216 years $(216-0.125)=215.875$.
2. Multiplication of 215.875 by $8=1727$.
3. Understanding that $(1727+1)=1728$, a number with a known integer cube root (12)
4. Reduction of the known integer cube root 12 by $1 / 2$ for the cube root of $216=6$.

## XVII. "Completing the cube" in astronomical context

Just how this connects with a "Point-Line-Square-Cube" analogy becomes apparent if the first number in VAT 8547 is the Triple Interval's 27-year period of revolution and the remaining periods in this text the sequential Fourth (64), Fifth (125) and Sixth Interval (216) in astronomical context.

| 1 <br> Point | Vine <br> Line $^{1}$ | $R$ <br> Square $^{2}$ | $T$ <br> Cube $^{3}$ | [ Unity, <br> Vi supplies the names for the | Velocity, |
| :---: | :---: | :---: | ---: | :--- | :--- | :--- |
| 1 | 3 | 9 | 27 | Distervals |  |

Table 7a. The Four Intervals in VAT 8547 in astronomical context.

Plus in addition, the inclusion of the Single [ $1,1,1,1$ ] and Double [ $1,2,4,8$ ] intervals serves to emphasize that the method extends from unity for all integers, while in passing providing (via the distances) the dimensions $1 \times 4 \times 9$, i.e., those of Sir Arthur C. Clarke's enigmatic, fictional 2001 monolith .. .

| 1 <br> Point | Vi <br> Line $^{1}$ | $R$ <br> Square $^{2}$ | $T$ <br> Cube | [ Unity, <br> Vi supplies the names for the Intervals |  |
| :---: | :---: | :---: | :---: | :--- | :--- | :--- |
| 1 | 1 | $\mathbf{1}$ | 1 | Single | Interval, ref. Vi, $R, T$. |
| 1 | $\mathbf{2}$ | $\mathbf{4}$ | 8 | Double | Interval (Binary) |
| 1 | 3 | $\mathbf{9}$ | 27 | Triple | Interval (Ternary) |

Table 7b. The Three initial Intervals in astronomical context.
However, an Old Babylonian text (CBS 8165) ${ }^{74}$ begins with data for these same three cubes described as follows by Neugebauer and Sachs (1945:34). The missing four cubes (out of seven) from VAT 8547) are suggested on the right.

No. 32. CBS 1865. Fragment of a single table. Obverse cube roots beginning with

$$
\begin{aligned}
1 \mathrm{e}-1 & \mathrm{ba}^{-s \mathrm{si}_{8}} \\
8 \mathrm{e}-1 & \mathrm{ba}_{\mathrm{s}-\mathrm{si}_{8}}
\end{aligned}
$$

Only parts of seven lines are preserved. The Reverse is inscribed with scattered, half-erased numbers which are obviously connected with the calculation of cubes, e.g., $2,13,20=\left(20^{3}\right)$ [emphases added).

No. 32. CBS 1865. single table, 1 through 216?
Obverse cube roots beginning at 1 with additions to 343, the septenary.

$$
\begin{array}{rll}
\text { 1e-1 } & \text { ba-si } & \text { Unity } \\
8 \mathrm{e}-1 & \mathrm{ba}-\mathrm{si}_{8} & \text { Double Interval } \\
27 \mathrm{e}-1 & \mathrm{ba}-\mathrm{si}_{8} & \text { Triple Interval } \\
64 \mathrm{e}-1 & \mathrm{ba}-\mathrm{si} \\
8
\end{array} \text { Quadruple Interval } 1 \text { Quad }
$$

Even so, apart from Earth, with the Double interval uncorrelated, the periods begin with that of the outermost known planet Saturn, and although the last period of 216 years includes the periods of Neptune (163.7232 years) and Uranus ( 83.7474 years), there is little to connect these values with the next three intervals. However, certain numerical properties for this set are known from later works, including the relation $a^{3}+b^{3}+c^{3}=d .^{3}$ Thus, for the sequence $27+64+125=216, a=3, b=4, c=5$ and $d=6$, which are the required cube roots of the four Intervals in question. Furthermore, it turns out that the distances for Uranus and Neptune are sequentially inherent in this set by way of the geometric means, initially from the products for the distances, e.g., between the Fourth and Fifth Intervals $(4 \cdot 5=20)$ for the distance $(R)$ of Uranus, and between the Fifth and Sixth Intervals $(5 \cdot 6=30)$ for the distance $(R)$ of Neptune. Both followed in turn by the square roots of the two distances for $V i$ to complete the geometric means with ( $T$ ), the corresponding periods obtainable from $V_{i}{ }^{3}$ or $V i \cdot V_{i}{ }^{2}=T$, etc,. But either way, note the 30 ("a. $u^{\prime \prime}$ ) for the distance of Neptune is the candidate for the "completing the square" exercise and the ellipse for this planet; likewise the 20 (a.u.) for Uranus. and the 9 (a.u.) for Saturn.

|  |  |  |  | [U |
| :---: | :---: | :---: | :---: | :---: |
|  |  | Square | $\mathrm{e}^{2}$ Cube ${ }^{3}$ | $V i$ |
| 1 |  | 1 |  |  |
| 1 | $\checkmark 2$ | 2 | $\checkmark 2.2$ | (Gm, $) T=2.828427$ ye |
| 1 | 2 | 4 | 8 | Double Interval, Binary. |
| 1 | $\checkmark 6$ | 6 | $\checkmark 6.6$ | (Gm), $T=14.69693$ years |
| 1 | 3 | 9 | 27 | Saturn, Triple Interval, Ter |
| 1 | $\checkmark 12$ | 12 | $\checkmark 12.12$ | (Gm), $T=41.56922$ years |
| 1 | 4 | 16 | 64 | Quadruple Interval, Quaternary. |
| 1 | $\checkmark 20$ | 20 | $\checkmark 20 \cdot 20$ | Uranus (Gm), $T=89.44272$ years |
| 1 | 5 | 25 | 125 | Quintuple interval, Quinary. |
| 1 | $\checkmark 30$ | 30 | $\checkmark 30 \cdot 30$ | Neptune (Gm), $T=164.3168$ y |
| 1 | 6 | 36 | 216 | Sextuple Interval, Senary. |

Table 7c. The Six integer Intervals \& five Geometric means.
Whereas at another time and place the fact that the Double [1,2,4,8] and Triple [1,3,9,27] Intervals predominate in ancient writings includes the condensed arrangement by Proclus in his Commentary on the Timaeus of Plato: ${ }^{75}$
"the terms $2,3,4,9,8$ and 27 with reference to the first part, ascribes the principal dignity to the monad."
also noteworthy for the positioning of 9 before 8 , but understandable with the inclusion of the monad $\{1\}$ as the standard frame of reference. Thus, although doubled-up, the following familiar arrangements:

| Point $^{1}$ | Vi <br> Line $^{1}$ | $R$ <br> Square | Cube <br> Cu |
| :---: | :---: | :---: | :---: |
| \{1\}, | $\mathbf{2 , 3 ,}$ | $\mathbf{4 , 9 ,}$ | $\mathbf{8 , 2 7}$ |

7d. Proclus: Combined Intervals.
with the "squaring and cubing" of $(V \mathrm{~V})=2$ and $(V \mathrm{Vi})=3$ providing the parameters for both sets of $(R)$ and $(T)$.
Lastly, there remains the place of the Double interval in all this, for apart from joining with the triple interval as the first integer sets that describe the harmonic law there is no planet per se that fits this first interval. Furthermore, the understanding that unity provides the frame of reference is also a necessity for this application. In this respect Macrobius (ca. 400 CE) in Commentary on the Dream of Ciprio, states: "The monad represents the point because, like the point, which is not a body but which produces bodies from itself." ${ }^{76}$ Emphasis on "bodies" may seem strange, but astronomical bodies and their distances fits well enough, especially since Macrobius, commencing with 2, next combines all the double and triple numbers to determine "solid bodies." This process includes the cubing of both 2 and 3, followed by the statement that: "In either case the monad is necessary to produce a solid body." Thus for the Double, Triple and all such intervals the monad (unity) serves as the first parameter and frame of reference, and the second (Vi) provides both the name and the number to be squared and cubed for mean distances and periods as stated earlier. Finally, for expository purposes at least, with the radius (or the heliocentric distance) of Earth 1, and diameter of 2, for the double interval [ $1,2,4,8$ ] the corresponding cube - length of side 2, area of one surface $4\left(\mathrm{Vi}^{2}\right)$ and volume of $8\left(V_{i}{ }^{3}\right)$ - are all realized. As, perhaps, similarly reflected in the following assignment by Philolaus:77

The number 8, which the arithmeticians call the first actual cube, has been given by the Pythagorean Philolaus the name of geometrical harmony, because he thinks he recognizes in it all the harmonic relations. and the following extended summary: ${ }^{78}$ (Cassidorus, Exp. in Ps., p. 36).

> [According to Pythagoras] . . Number is the first principle, a thing which is undefined, incomprehensible, having in itself all numbers which could reach infinity in amount. And the first principle of numbers is in substance the first Monad, which is a male monad, begetting as a father all other numbers. Secondly, the Dyad is a female number, and the same is called by the arithmeticians even. Thirdly, the Triad is a male number, this the arithmeticians have been wont to call odd. Finally, the Tetrad is a female number, and the same is called even because it is female..... Pythagoras said this sacred Tektractys is: 'the spring having the roots of ever-flowing nature.' ... the four parts of the Decad, this perfect number, are called number, monad, power and cube.
> (Hippol., Phil,. 2. Dox. 355).

## XVIII. The Pythagorean Tetractys and the point-line-area-volume analogy

However, it becomes apparent that the "Double" and "Triple" intervals in Plato's Timaeus and the point, line, area, volume analogy are also linked to the Pythagorean Tetractys. Which can now be considered in terms of the "lines" and the "areas" in the present "completing the square" exercise. In fact, the first line "I have subtracted the side of my square from the area" leads directly to the groupings of four in the Pythagorean Tetractys, beginning simply enough with the first four integers as expanded further in the following commentary by Thomas Taylor: ${ }^{79}$

The first is that which subsists according to the composition of numbers.
The second, according to the multiplication of numbers.
The third subsists according to magnitude.
The fourth is of the simple bodies.
The fifth is of figures.
The sixth is of things rising into existence through the vegetative life.
The seventh is of communities.
The eighth is the judicial power.
The ninth is of the parts of the animal.
The tenth is of the seasons of the year.
And the eleventh is of the ages of man.
All of them however are proportional to each other. For what the monad is in the first and second tetractys, that a point is in the third; fire in the fourth; a pyramid in the fifth; seed in the sixth; man in the seventh; intellect in in the eighth; and so of the rest. Thus, for instance, the first tetractys is 1.2.3.4. The second is the monad, a side, a square, and a cube. The third is a point, a line, a superficies, and a solid. The fourth is fire, air, water, earth. The fifth the pyramid, the octahedron, the icosahedron, and the cube. The sixth, seed, length, breadth and depth. The seventh, man, a house, a street, a city. The eighth, intellect, science, opinion, sense. The ninth, the rational, the irascible, and the epithymetic parts, and the body. The tenth, the spring, summer, autumn, winter. Eleventh, the infant, the lad, the man, and the old man.

The world also, which is composed from these tetractys, is perfect, being elegantly arranged in geometrical, harmonical, and arithmetical proportion; comprehending every power, all the nature of number, every magnitude, and every simple and composite body. But it is perfect, because all things are the parts of it, but it is not itself the part of any thing. Hence, the Pythagoreans are said to have first used the before-mentioned oath, and also the assertion that "all things are assimilated to number."

Thomas Taylor, IAMBLICUS' LIFE OF PYTHAGORAS

As for the formulation of the harmonic law in terms of points, lines, areas and volumes, this is at least systematic in contrast to the routine acceptance of the surprising fact that in the Solar System the cube of a radius vector is equal to the square of the time the latter takes to complete $360^{\circ}$ about the "center." Small wonder, then, that one finds included in ancient writings, e.g., Plato's Epinomis, the following condensed commentary concerning: ${ }^{80}$
.. what is called by the very ludicrous name mensuration, but is really a manifest assimilation to one another of numbers which are naturally dissimilar, effected by reference to areas. Now to a man who can comprehend this, it will be plain that this is no mere feat of human skill, but a miracle of God's contrivance. Next, numbers raised to the third power and thus presenting an analogy with three-dimensional things. Here again he assimilates the dissimilar by a second science, which those who hit on the discovery have named stereometry [the gauging of solids], a device of God's contriving which breeds amazement in those who fix their gaze on it and consider how universal nature molds form and type by the constant revolution of potency and its converse about the double in the various progressions. The first example of this ratio of the double in the advancing number series is that of 1 to 2 ; double of this is the ratio of their second powers [1:4], and double of this again the advance to the solid and tangible, as we proceed from 1 to $8\left[1,2,2^{2}, 2^{3}\right]$; the advance to a mean of the double, that mean which is equidistant from lesser and greater term [the arithmetical], or the other mean [the harmonic] which exceeds the one term and is itself exceeded by the other by the same fraction of the respective terms- these ratios of $3: 2$ and $4: 3$ will be found as means between 6 and 12-why, in the potency of the mean between these terms [ 6,12 ], with its double sense, we have a gift from the blessed choir of the Muses to which mankind owes the boon of the play of consonance and measure, with all they contribute to rhythm and melody. So much, then, for our program as a whole. But to crown it all, we must go on to the generation of things divine, the fairest and most heavenly spectacle God has vouchsafed to the eye of man.

PLATO, Epinomis 990c -991b, trans. A. E Taylor (1531-1532)
All of which sheds some light on the emphasis placed on the latter, the triple interval and their relationship to the Timaeus, as Macrobius explains at the end of a lengthy passage after introducing the number 7 alone and also in association with 8 . The passage concludes by assigning an all-encompassing significance to the number 5 while also providing points of immediate relevance with a hint of more to follow: ${ }^{81}$

## [ Chapter V, Macrobius, Commentary on the Dream of Scipio. ]

[12] Thus it becomes clear that numbers precede surfaces and lines (of which surfaces consist), and in fact come before all physical objects. From lines we progress to numbers, to something more essential, as it were, so that from the various numbers of lines we understand what geometrical figures are being represented.
[13] But we have already remarked that surfaces with their lines are the first incorporeality after the corporeality of bodies and that they are nevertheless not to be separated from bodies on account of their indissoluble union with them. Therefore, whatever precedes surface is purely incorporeal; but we have shown that number is prior to surface and to lines; hence the first perfection of incorporeality is in numbers, and this is, as we previously stated, the common perfection and fullness of all numbers.
[14] The fullness of those numbers which form bodies or bind them together is a particular one, as we suggested above and will explain shortly; at the same time I shall not deny that there are other reasons for numbers being full.
[15] That the number eight produces a solid body has been demonstrated above. But this number has a special right to be called full, for in addition to its producing solid bodies it is also without doubt intimately related to the harmony of the spheres, since the revolving spheres are eight in number. More about this later, however.
[16] All numbers which, when paired, total eight are such that fullness is produced from their union. The number eight is either the sum of the two numbers that are neither begotten nor beget, namely, one and seven, whose qualities will be discussed more fully in their proper place; or from the doubling of that number which is both begotten and begets, namely, four, since four comes from two and produces eight; or from three and five, one of which is the first uneven number of all, while the characteristics of the other will be treated later. [End of Chap. V].

Chapter Vi, Macrobius, Commentary on the Dream of Scipio. ${ }^{82}$
[1] FURTHERMORE, a reason patent to all persuades us that the number seven also deserves to be called full. But we cannot pass over this fact without first expressing admiration that of the two numbers which, when multiplied with each other, determine the life span of the courageous Scipio, the one is even, the other odd. Indeed, that is truly perfect which is begotten from a union of these numbers. An odd number is called male and an even female; mathematicians, moreover, honor odd numbers with the name Father and even numbers with the name Mother.
[2] Hence Timaeus, in Plato's dialogue by the same name, says the God who made the WorldSoul intertwined odd and even in its make-up: that is, using the numbers two and three as a basis, he alternated the odd and even numbers from two to eight and from three to twenty-seven.
[3] The first cubes in either series arise from these: using the even numbers, two times two, or four, make a surface, and two times two times two, or eight, make a solid; again, using the odd numbers, thrice three, or nine, make a surface, and three times three times three, or twenty-seven, the first cube.

Chapter Vi, Macrobius, Commentary on the Dream of Scipio. (continued).
Accordingly we are given to understand that these two numbers, I mean seven and eight, which combine to make up the life-span of a consummate statesman, have alone been judged suitable for producing the World-Soul for there can be no higher perfection than the Creator.
[4] This, too, must be kept in mind, that in affirming the dignity belonging to all numbers we showed that they were prior to surfaces, lines, and all bodies; and besides we learned a moment ago that numbers preceded the World-Soul, being interwoven in it, according to the majestic account in the Timaeus, which understood and expounded Nature herself.
[5] Hence the fact which wise men have not hesitated to proclaim is true, that the soul is a number moving itself. Now let us see why the number seven deserves to be considered full on its own merits. That its fullness may be more clearly realized, let us first examine the merits of the numbers whose sums make up seven, then, at last, the capabilities of seven itself.
[6] The number seven is made up either of one and six, two and five, or three and four. It would be well to treat these combinations separately; we will confess that no other number has such a fruitful variety of powers. (Arguments [7] through [18] omitted)
[19] The possession of unusual powers came to the number five because it alone embraces all things that are and seem to be. (We speak of things intelligible as "being," and of things material as "seeming to be," whether they have a divine or a mortal body.)" Consequently, this number designates at once all things in the higher and lower realms. (emphases supplied)

Macrobius, Commentary on the Dream of Scipio, Chapter Vi, Trans by William Harris Stahl. (emphases added)
In point [ 2 ] of Chapter Vi, the use of "the numbers two and three as a basis" to construct the "World-Soul" in Plato's Timaeus is by now familiar in so much as it incorporates the single line of Proclus (Fig. 4d) :
[2] Hence Timaeus, in Plato's dialogue by the same name, says the God who made the World-Soul intertwined odd and even in its make-up: that is, using the numbers two and three as a basis, he alternated the odd and even numbers from two to eight and from three to twenty-seven.
with later statements by Macrobius which refer to seven, a number, which also "deserves to be considered full on its own merits" ( [5] ).

Further, the number seven is also included as the sum of one and six, two and five, plus (finally) three and four, followed by the injunction that "it would be well to treat these combinations separately; we will confess that no other number has such a fruitful variety of powers." ( [6] ). Grouped in this manner it is difficult not to suspect that the Lucas series, i.e., $1,3,4,7, \ldots$ is part of the dialogue, except, of course, that Francois Lucas (1842-1891) is credited with the (belated) "discovery" of this elementary series.
As for the bald and definite statement in [3] that the numbers seven and eight "combine to make up the life-span of a consumate statesman (and) have alone been judged suitable for producing the World-Soul," this may, perhaps, refer to Plato's Statesman. The latter begins with a laborious emphasis on the number three, and later seeks "to define a perfect constitution" from "the rule of one, the rule of the few, and the rule of the many." This is followed by somewhat obscure instructions; "Dividing each of the three into to two let us make six, having first separated the true constitution from all, calling it the seventh," noting next that "Under the rule of one we get kingly rule and tyranny; under the rule of the few, as we said, come the auspicious form of if, aristocracy, and also oligarchy." ${ }^{83}$
It is at this point that the "Number of the Tyrant" in Plato's Republic may be considered, with both the latter and the present topic treated in terms of the eighth Pythagorean tetractys - "judicial power." Here, according to Macrobius, the number of the (consummate) statesman is explicitly defined as the product of 7 and $8=56$ years.
This period is readily associated with the second perfect number, 28 , or more precisely, twice this value (56). This is an elementary matter, to be sure, but 28 years lies well within the range of periods provided by the apsidal data for Saturn (approximately 27 to 32 years) and also, with 2 periods of revolution for Saturn in 56 years, there will be 3 synodic lap cycles (SD1) and 5 revolutions for Jupiter to complete the ( $2: 3: 5$ ) resonant triple for these two major planets. Furthermore, continuing in the same vein, comprehension is gained from the understanding that the base period of the Pierce planetary framework (and/or its fore-runners) will be 6 times that of the period of Saturn.
Therefore the required base period $P 1$ is the product of the first and second perfect numbers ( $6 \cdot 28=168$ years) which can then be applied in standard fashion to the Pierce divisors shown below in Table 5.
Here, however, the correlation with the Phi-series begins to deteriorate beyond Jupiter but continues to improve below this planet en route to Mercury:

| PLANETS N Synodics \# | RATIOS <br> (Pierce) | DIVISORS (added) | $\begin{gathered} \text { RES.TRIPLES } \\ {[(\text { RZT) }]} \end{gathered}$ | PERIODS (Days) P1/Divisors (JYR) | PERIODS (Years) P1/Divisors (JYR) | $\begin{gathered} \text { Phi-series } T={ }^{\mathrm{x}} \\ \text { Exponents: } \end{gathered}$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Neptune 1 | 1/1 | 1 |  | 61,362 | 168 |  |  |
| Synodic 2-1 | 1/1 | 1 | 1(1)2 | 61,362 | 168 |  |  |
| Uranus 2 | 1/2 | 2 |  | 30,681 | 84 |  |  |
| Synodic 3-2 | 1/2 | 4 | 1(2)3 | 15,340.5 | 42 |  |  |
| Saturn 3 | 2/3 | 6 |  | 10,227 | 28 | 29.03444185 | 7 |
| Synodic 4-3 | 2/3 | 9 | 2(3)5 | 6,818 | 18.666* | 17.94427191 | 6 |
| Jupiter 4 | 3/5 | 15 |  | 4,090.8 | 11.2 | 11.09016994 | 5 |
| Synodic 5-4 | 3/5 | 25 | 3(5)8 | 2,454.48 | 6.72 | 6.854101966 | 4 |
| M-J Gap 5 | 5/8 | 40 |  | 1,534.05 | 4.2 | 4.236067977 | 3 |
| Synodic 6-5 | 5/8 | 64 | 5(8)13 | 958.78125 | 2.625 | 2.618033989 | 2 |
| Mars 6 | 8/13 | 104 |  | 590.01923 | 1.6153846154 | 1.618033989 | 1 |
| Earth/Syn 7-6 | 8/13 | 169 | 8(13)21 | 363.08876 | 0.9940828402 | 1.000000000 | 0 |
| Venus 7 | 13/21 | 273 |  | 224.76923 | 0.6153846154 | 0.618033989 | -1 |
| Synodic 8-7 | 13/21 | 441 | 13(21)34 | 139.14286 | 0.3809523809 | 0.381966011 | -2 |
| Mercury 8 | 21/34 | 714 |  | 85.911765 | 0.2352941176 | 0.236067978 | -3 |

Table 8. Pierce planetary framework, Solar System \& Phi-series. Base period from perfect numbers $6 \cdot 28=168$ years.

## IXX. More on the Triple Interval and mathematical astronomy in Plato's Dialogues

The process of squaring and cubing to determine distances and periods is not only simpler than using fractional exponents, it also lends itself to easy dissemination, especially the integer parameters associated with the triple interval, e.g., [1, 3, 9, 27]: Unity (1), Velocity Vi (3), Distance R (9) and corresponding period at perihelion (27 years) as applied to Saturn, the outermost planet known in Antiquity. Which in this form plays a role in familiarization as exemplified by the condensed details in Plato's Republic $I X, 587 \mathrm{~d}$-588a. ${ }^{84}$ "Three times three, then, by numerical measure," that "by longitudinal mensuration (is) a plane number," followed "by squaring and cubing," after which "it is clear what the interval of this separation becomes." Also, by "taking it the other way about," and if one "tries to express the extent of the interval .... he will find on completion of the multiplication that he lives 729 times as happily and that the tyrant's life is more painful by the same distance." Understandably, these lines represent "An overwhelming and baffling calculation ... and what is more, it is a true number and pertinent to the lives of men if days and nights and months and years pertain to them." Thus not only the same process, but also, it would seem, the triple interval with both time and distance incorporated in the dialogue, albeit obliquely. Here, commencing with $V i^{2}$ the distance 9 (a.u.) is followed by "squaring and cubing" and the common product 729 ( $T^{2}=27^{2}$ and $R^{3}$ $=9^{3}$ ) included for good measure. On the other hand, the confusion caused by references to: "life," "the tyrant" and "pain and pleasure" in the above become somewhat more understandable when the methodology inherent in the Pythagorean Tetractys is taken into consideration.

But above all else, it is the suitability of the Triple interval [1, 3, 9, 27] - Ref. Unity, velocity Vi = 3, Distance 9 (a.u) and corresponding period of revolution of 27 years applied to Saturn which is the most helpful. Then again, the data pertain to the shortest period (27 years) associated with perihelion, and not the mean period or longest period at aphelion, even though the latter was the result of the "completing the square" approach to elliptical orbits. Nor for that matter does rounding the aphelion period to 32 years help greatly either since the distance 10 (a.u.) now becomes 10.079368 (a.u.). And also, multiplication of the velocity (3.162277) by 10 , the corresponding distance to produce the period $T$ of 31.622777 years is also lost. Although not lost entirely, since in condensed form it is again relation (4) from the 1989 Galileo paper discussed earlier, i.e., $T=V i^{3}$.

| APSIDES | Unity | Velocity Vi | Distance R | Period T |
| :--- | :---: | :---: | :---: | :---: |
| Perihelion | $\mathbf{1}$ | $\mathbf{3}$ | $\mathbf{9}$ | $\mathbf{2 7}$ |
| SATURN | 1 | 3.0822070 | 9.5 | 29.280967 |
| Aphelion | 1 | 3.1622777 | 10 | 31.622777 |
| Perihelion | $\mathbf{1}$ | $\mathbf{3}$ | $\mathbf{9}$ | 27 |
| SATURN 2 | 1 | 3.0898733 | 9.547317 | 29.5 |
| Aphelion | 1 | 3.1748021 | 10.079368 | 32 |

Table 9. The Triple Interval and integer values

Whereas the 32-year aphelion period assigned to Saturn reappears among the workings of the "Alchemists" in the far from straightforward context supplied below. Although still a difficult text, towards the end some degree familiarity with material presented here in the preceding pages becomes apparent. In particular, the question:

> "If the Capacity or Circumference of the sphere be 32 foot, how much will one of the sides of the Cube be to Equalize the Capacity of this Sphere?"
at least begins to swing the dialogue towards the "Point-Line-Area-Volume" analogy, "completing the cube," and a reference to Saturn, plus "six the first of the perfect numbers."

## Atalanta fugiens emblems 21-25

Michael Maier's alchemical emblem book Atalanta fugiens was first published in Latin in 1617. It was a most amazing book as it incorporated 50 emblems with epigrams and a discourse, but extended the concept of an emblem book by incorporating 50 pieces of music the 'fugues' or canons. In this sense it was an early example of multimedia. ${ }^{85}$

## Emblem XXI. [transcribed by Hereward Tilton] <br> Make of the man and woman a Circle, of that a Quadrangle, of this a Triangle, of the same a Circle and you will have the Stone of the Philosophers.

[ Closing paragraphs ]
But that this was not unknown to the Philosophers of Nature is apparent from this: That they command a Circle to be turned into a Quadrangle, and this by a Triangle to be reduced again to a Circle. By a circle they understand the most simple body without angles, as by the Quadrangle they do the four Elements. It is as if they should say: The most simple corporeal Figure that can be found is to be taken and divided into four Elementall Colours, becoming an Equilaterall Quadrangle. Now every man understands that this Quadration is Physicall and agreeable to Nature, by which far more benefit accrues to the Publick, and more light appears to the mind of Man, than by any meere Theory of Mathematicks when abstracted from Matter. To learn this perfectly a Geometrician acting upon solid bodyes must enquire what is the depth of solid Figures, as for example the Profundity of Sphere and Cube must be knowne and transferred to manuall use and practice. If the Capacity or Circumference of the sphere be 32 foot, how much will one of the sides of the Cube be to Equalize the Capacity of this Sphere? On the contrary, one might look back from the Measures which the Cube contains to the feet of each Circumference.
In like manner the Philosophers would have the Quadrangle reduced into a Triangle, that is, into a Body, Spirit and Soul, which three appear in the three previous colours before Rednesse: that is, the Body or earth in the Blacknesse of Saturn, the Spirit in the Lunar whitenesse as water, and the Soul or air in the Solar Citrinity. Then the Triangle will be perfect, but this again must be changed into a Circle; that is, into an invariable rednesse, by which operation the woman is converted into the man and made one with him, and six the first of the perfect numbers is absolved by one, two having returned again to an unity in which there is Rest and eternall peace.

Closing paragraphs for Atalanta fugiens emblem 21 by Michael Maier (1617). Introduced by Adam Mclean. ${ }^{85}$
The cryptic description of Emblem XXI "Make of the man and woman a Circle, of that a Quadrangle, of this a Triangle, of the same a Circle and you will have the Stone of the Philosophers" is discussed in Part Four with respect to isosceles and equi-lateral triangles emphasized in Plato's Timaeus. Linked to the "Rotation of the Elements," it is shown that the former pertain to the Fibonacci series with ratios between the sides of the triangles increasing towards Phi itself as the rotations continue. The Lucas series variant makes use of the same rotations but differs in range and purpose as it continues to approximate the $1 / 2$ Phi-series, improving towards this outcome with each successive rotation.

## XX. The Body, Spirit and Soul Triad

But there is also something else of importance, which is the reference to "a triangle" reduced into "a Body, Spirit and Soul." Of the three, "body" is perhaps the simplest since it is a well attested term for an astronomical body. Furthermore, with this key providing the way, there are sufficient examples in the literature to also assign "spirit" to Velocities ( $V_{i} \& V_{r}$ ) and "soul" to Time (T,S), but not without some difficulty in the absence of additional information. In terms of the extant alchemical literature, Philip à Gabella (1615) perhaps leads the way by passing on that: ${ }^{\text {s6 }}$ according to the first fathers of philosophy the magical contemplation of the ternary encompassed body, spirit and soul while in more detail, Johann Isaac Hollandus in Of natural \& supernatural things. London, 1670, writes: ${ }^{87}$

They say, in our Stone are the four Elements, and they say true; for the four Elements must be separated out of Saturn. They say, our Stone consists of Soul, Spirit and Body, and these three become one. They say true; when it is made fixed for the white Mercury and Sulphur with its' Earth, then these three are one.
and in Paracelsus his Aurora, \& Treasure of the Philosophers (1659) where "Mercurius" states in part: ${ }^{\text {88 }}$
"This mystery it is permitted only to the prophets of God to know. Hence it comes to pass that this Stone is called animal, because in its blood a soul lies hid. It is likewise composed of body, spirit, and soul. For the same reason they called it their microcosm, because it has the likeness of all things in the world, and thence they termed it animal, as Plato named the great world an animal."
plus more from The naturall Chymicall Symboll or short Confession of Doctor Kunwrath: ${ }^{89}$
[Three in One, One in Three]. But this is the true philosophicall doctrine of the philosophers Mercury, That Three is One generall Chaos, Three in essence, namely Body, Soule, and Spiritt; and these Three Essences are had in One substance or thing and neere at hand.
[Body Soule Spirit.] There is one Essence of the Body one other of ye Spirit, one other of the Soule; But ye Body, Soule, and Spirit are one thing, wherein all the three are together equally necessarily present at the same time.

For the Body is not made by the Arte of Man, nor is the Spirit made by the Arte of Man, neither is the Soule made by the Arte of Man.
(all emphases supplied)
Macrobius, on the other hand, in his Commentary on the Dream of Scipio is more direct while at the same time invoking the Double and Triple intervals with emphasis on assigning the last exponent of the Point-Line-square cube analogy to a solid body, then finally arriving at Soul = time and motion "with its animating power" and non-corporeal form: ${ }^{90}$
[12] Since the monad is the source of even and uneven numbers alike, the number three should be considered the first line. This tripled gives nine, which from its two lines, as it were, produces a body with length and breadth, as was the case with the number four, the second of the even numbers. In the same way the number nine tripled supplies the third dimension. Thus with the uneven numbers a solid body is formed in twenty-seven, three times three times three, just as with even numbers eight or two times two times two made a solid body.
[13] In either case the monad is necessary to produce a solid body, in addition to the other six numbers, three even and three uneven, two, four, and eight being the even numbers and three, nine, and twenty-seven the uneven.
[14] Plato's Timaeus, in disclosing the divine plan in the creation of the World-Soul, said that the Soul was interwoven with those numbers, odd or even, which produce the cube or solid, not meaning by this that the Soul was at all corporeal; rather, in order to be able to penetrate the whole world with its animating power and fill the solid body of the universe, the Soul was constructed from the numbers denoting solidity.
(all emphases supplied)

A small selection, to be sure and also not particularly compelling, but the complexity of the matter can at least be given a detailed basis in the form of the Phi-series planetary framework generated earlier in Table 3. As shown below, a reduced, rearranged version of the latter has double or triple occurrences of parameters in each of the three categories - Soul (Periods $T$ ), Body (Distances $R$ ) and Spirit (Velocity $V i$ ) with ( $V r$ ) not part of the set but available:

|  | Unity |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Synodics \# | Po |  |  |  |  | Vi) |
| urn 3 | 1 | 3.073532624 | . 446602789 | 29.03444185 |  | 0.325358512 |
| Synodic 4-3 | 1 | 2.618033 | 966 | 1 | 6 | 011 |
| Jupiter 4 | 1 | 2.23004041 | . 97308025 | 11.09016994 | 5 | 0.448422366 |
| Synodic 5-4 | 1 | 1.899547627 | 3.608281187 | 6.854101966 | 4 | 0.526441130 |
| M-J Gap 5 | 1 | 1.618033989 | 2.618033989 | 4.236067977 | 3 | 0.618033989 |
| Synodic 6-5 | 1 | 1.37824077 | 1.899547627 | 2.618033989 | 2 | 0.725562630 |
| Mars 6 | 1 | 1.17398499 | 1.37824077 | 618033989 |  | 0.851799642 |
| Earth/Syn 7-6 | 1 | 1.000000000 | 1.000000000 | 1.000000000 | 0 | 1.000000000 |
| Venus 7 | 1 | 0.851799642 | 0.725562630 | 0.618033989 | 1 | 1.173984997 |
| Synodic 8-7 | 1 | 0.725562630 | 0.526441130 | 0.381966011 | -2 | 1.378240772 |
| Mercury 8 | 1 | 0.618033989 | 0.381966011 | 0.236067978 | -3 | 1.61803398 |

Table 3s. Phi-series Vr, Vi, R, T and the Point-Line-Square-Cube / Spirit-Body-Soul assignments.

## XXI. The World-Machine of Francesgo Giorgi (1525).

The significance of certain sets of data is more apparent for some than others; for example, the primary numbers for the Synodic $6-5$ position are, from right to left the three constants assigned to the mean planetary Periods, i.e., relation (8), the Distances of the Planets, relation (9) and the Periods for the planet-synodic expansions, relation (12). Other assignments of interest include the three locations for Phi itself, firstly as the Period (SOUL) of Mars, secondly as the Velocity Vr of Mercury and thirdly also the Velocity (Vi) (SPIRIT) for the Mars-Jupiter Gap. Whereas the three occurrences for the synodic 8-7 period $T$ (SOUL) between Venus and Mercury are the Distance $R$ (BODY) for Mercury, and again the Velocity (Vr) for synodic4-3 between Jupiter and Saturn. Which, in addition to dynamic aspects, also involves the first of unities" ("384") in a summation concerning the construction of a "world-machine" by Francesco Giorgi (1466-1540) whose Harmonia Mundi (1525) concludes with a remarkably condensed statement apparently concerned with the construction of a "world-machine" as opposed to the "world-soul" of Plato: ${ }^{91}$

[^4]The statement is, however, readily decoded, especially if the reader is familiar with both the "Ternary" and "Senary" (the Triple interval [1, 3, 9, 27] and the Sextuple interval [1, 6, 36, 216] respectively) in the present context. This said, it appears that there are two significant activities involved, firstly with the Phi-series concerning the number "27" and the range "from 384 to 10,368 " which concerns heliocentric planetary distances. And secondly, a separate approach which involves providing the upper limit for the outermost planet ("Neptune") of 162 (years) as given.

Remaining with the latter there is therefore now a further base period for the Pierce planetary framework and its divisors. Although the resulting period for Mercury is low (162/714 $=0.227209$ years) the division by 6 for Saturn, the outermost known planet in Antiquity (162/6) results in 27 years, which is also the period/volume parameter (or Soul again in years) of the Ternary. Whereas the product: $27 \cdot 384=10,368$, the range assigned in the text. Except for one thing, which is to provide a meaning for this data the range is best understood to extend from 0.384 "a.u." to 10.368 "a.u." in 27 steps. As for the hows and the whys of this matter, both are influenced by an earlier 36-step sequence, which is fortunately explained in the meticulous footnotes accompanying another relatively obscure work, this time by George Burges entitled: THE TREATISE OF TIMEUS THE LOCRIAN ON THE SOUL OF THE WORLD AND NATURE (London,1876).

Here Burges explains the relationship (and more) between the 36-step and 27-step configurations augmented by recognizable references to the Binary [ $1,2,4,8$ ], Ternary [ $1,3,9,27$ ] and the point-line-square-cube analogy. Plus something else, which is an extension to incorporate the "Soul of the World." ${ }^{92}$
> ... But why were the terms fixed at 36 ? The reason is to be found in the mysteries of the school of Pythagoras, where it was thought proper to multiply 384, the assumed term. By 27. But why 27 ? Because that number is the sum of the first numbers, which must represent lines, surfaces, solids, squares, and cubes, added to unity. Thus 1 is unity ; 2 and 3 , the first numbers representing lines : 4 and 9 , the first surfaces, and both squares, the former of an even number (2), and the latter of an odd number (3). Taking then the number 27 as the symbol of the world, and the numbers which it contains as the symbols of the elements and their combinations, it was reasonable for the Soul of the world, which is the very basis of order and of combinations, which constitute the world, to be composed of the same elements (of order) as the number 27 in itself.

> George Burges, Supplement to TIMEUS THE LOCRIAN (1876: 172-173)

This leaves the matter of the "First of Unities" - ("384") also discussed by Burges in TIMEUS THE LOCRIAN, The Works of Plato : A New Literal Version Vol. VI. (1876: 150-151), which in a technical, dynamic sense turns out to be far from simple. It is also, as a variation of other interpretations of Plato's Timaeus, impressive in the manner by which the "Body" = Distance (Square), and "Soul" = Period (Cube) methodology discussed earlier is extended and linked to the "First of Unities." ${ }^{93}$

> But the soul of the world has (the deity) united with the centre and led it outwards, investing the world wholly with it, and making it a mixture of Form undivided, and of Substance divided, so as to become one mixture from these two; for which (world) he mixed up two threes, the origin of motion, one connected with the same, the other with the different; which (soul), being mixed with difficulty, was mixed not in the easiest way. Now all these proportions are combined harmonically according to numbers: which proportions he has divided according to a scale scientifically, so that a person is not ignorant of what things and by what means the soul is combined; which the deity has not ranked after the substance of the body,- for, as we say, that which is before is in greater honor as regards both power and time, - but he made it older by taking the first of unities, which is 384.

George Burges, TIMEUS THE LOCRIAN (1876 : 150-151)
To understand the First of Unities it is sufficient to know that for the Phi-series the "Body," or heliocentric distance of Mercury is equal to the "SOUL," or period of the Mercury lap-cycle with respect to Venus, i.e., Synodic 8-7. Moreover,

| PLANETS N Synodics | Series | SOUL (T) Period/Time | $\begin{aligned} & \text { BODY }(R) \\ & \text { Distance (a.u.) } \end{aligned}$ | SPIRIT (Vi) Velocity | Reference Unity |
| :---: | :---: | :---: | :---: | :---: | :---: |
| NAMES \# | X | Cube, Vi ${ }^{3}$ | Square, Vi ${ }^{2}$ | Line, Vi ${ }^{1}$ | Unity Vi ${ }^{0}$ |
| Venus 7 | -1 | 0.618033989 | 0.725562630 | 0.851799642 | 2 |
| Synodic 8-7 | -2 | 0.381966011 | 0.526441130 | 0.725562630 | O |
| Mercury 8 | -3 | 0.236067978 | 0.381966011 | 0.618033989 | 9 |

Table 3tr. Synodic position 8-7. Phi-series T, S and Vi, Mercury through Venus.
in the same wise, this number is therefore 0.384 in both applications, i.e., 0.384 a.u. for the distance and 0.384 years for the period. Which then results in the range from 0.384 to 10.368 , with the latter the aphelion distance of Saturn as opposed to the lesser perihelion and mean distances. But even so, the increment for the 27 steps obtained from $(10.368-0.384) / 27=9.984 / 27$ is $0.369777^{*}$ with the sequence ending at 33.384316 years for the distance (BODY, R) of 10.368 a.u. at the 27 th step.

The interplay between "Body" and "Soul" by Burges in this passage plus the reference to "power and time" and the odd statement that the Creator "made it older by taking the first of unities, which is 384," is one method of placing the second instance of the parameter which occurs as the Distance of Mercury in the location of a second category. Thus also now associated with Time (Soul) and the "older" Mercury-Venus lap-cycle, Synodic 8-7 while utilising the the pheidian exponents $-1,-2,-3$ for the period ( $T, S$ ) for Venus ( $T$ ), Mercury-Venus Synodic 8-7 ( $S$ ) and Mercury ( $T$ ).

The reason for this odd duality - at least from a bare numerical viewpoint - is straightforward enough, simply the consequence of the Phl-series applied sequentially downwards to produce the periods $(T, S)$ and then using Kepler's Third Law of planetary motion to determine the Distance of Mercury. Thus, for the Phi-Series, $\left.{ }^{x}, x=-1-2,-3, \ldots n\right)$ first the period of revolution of Venus $=^{-1}$, next the period for Synodic $8-7={ }^{-2}$ and last, the period of Mercury $={ }^{-3}$. However, in exponential terms, applying the conversion from Period to Distance, i.e., $\mathrm{R}=\mathrm{T}^{2 / 3}$ the latter applied to Mercury ${ }^{-3}$ results in $\left({ }^{-3}\right)^{2 / 3}={ }^{-2}$, which is identical to the exponent which generates Synodic 8-7 and the duality.

## Remarks

Just when and how this situation came to be uncovered is difficult to say, of course. Perhaps it was Pythagorean, perhaps earlier, or perhaps a little later. But in any event, there are increasingly more questions than answers at the present time, and it is likely that there are more aspects buried in the Tetractys which may have roles to play. For example, further divisions arising from Odd and Even numbers and their Male and Female assignments, the social, legal and political aspects, and the complexities embraced by factoring in "Life" and "Nature" itself.

Certainly among the alchemists various bits and pieces are discernable; witness now a partial agreement with the last remark above by Marcilio Ficino (1518), who states: ${ }^{94}$

For although sulphur and Mercury were as it were the root of metals before the first coagulation, yet now they are not, since they are brought to another nature: whence it remains that there cannot be made out of them any metallic body. Since also the chain is unknown, by which Venus \& Mercury copulated together in due proportion.

Marsilius Ficinus, 'Liber de Arte Chemica', in Theatrum Chemicum, Vol 2, 1702.
thus a "hermaphroditic" component associated with the adjacent positions for Synodic 8-7 and Mercury a further complication. Here, however, in an extract from Bernard of Trevisan, Le Texte d'Alchymie et le Songe-Verd, (1695) a slightly more understandable account is offered: ${ }^{95}$

Therefore, my child, you see very well that I have declared all to you when I have made you understand in what manner our Sulfur is contained in the belly of the Mercury, and that it is correct to call it internal Sulfur or hidden Spirit, which is no other thing than heat and dryness, acting on the cold and the moisture, acting on the patient, the pure mercurial substance of which Sulfur is the Soul, since it is it which vivifies and sustains the Mercury which would be, without our Sulfur, only a dead, unfruitful, and sterile earth. There is then good reason to say that Sulfur and Mercury are the proper and true substances of the metals, because it is very certain that this Sulfur cannot be without Mercury and that our Mercury cannot be without this Sulfur which is intimately united and incorporated with it, as the soul is with the body.
These two names of Mercury and of Sulfur are only names for one single substance which we know under the names of Quicksilver or Mercury.
Therefore make manifest that which is hidden and make occult that which is manifest. I tell you, in that alone consists the work of the sages. Our gum curdles our milk, and our milk dissolves our gum, and they grow in the Stone of Paradise, which Stone is of two contrary natures, that is to say, of the natures of Fire and of Water.
All that I have written above ought to have opened your understanding to the intelligence of the philosophers - for what I have explained to you altogether well and have given you to understand what our Sulphur is, that the philosophers have also called Gum, Oil, Sun, Fixity, Red Stone, Curd, Safran, Poppy, Red Brass, Tincture, Dry, Fire, Spirit, Agent, Soul, Blood, Burned Brass, Red Man, and Quick Earth. I have also given you a clear and concise explanation of that which the philosophers name Water, Milk, White Wrapper, White Manna, White Urine, Cold, Moisture which does not dampen, Body, Womb, Moon, White Woman, Changing Habit, volatile, patient, Virginal Milk, Lead, Glass, White Flower, Flower of Salt, Fleece, Veil, Venom, Alum, Vitriol, Air, Wind, Rainbow, Naked Woman, and so many other names which are only for the purpose of making us conceive the qualities, properties, and the two natures of male and female contained in our substance, which is nothing else but animated Quicksilver. It is this viscous moisture mixed with its earthy part, our Mercury, and the true foundation of all our science.
It is in this great number of terms that the wise men have taken pleasure in writing their sentiment relative to our science. All these names ought to convince you of the truth of our science, for all of them have only one meaning and all of them have for their purpose only to expose the hermaphroditic Mercury to us. It is feminine if it is considered as separated from the Sulfur which it contains within it and of which it is the substance; but it is masculine if it is considered according to its Sulfur with which it is united so intimately that it cannot be separated from it; and it can be said of their marriage that they are both of them in the same flesh.

Bernard of Trevisan, Le Texte d'Alchymie et le Songe-Verd (1695).

But there is something else which intrudes, for in addition to the mention of Mercury and sulphur (the latter called here the soul) is what can be understood to be the third occurrence of 0.384 as an "internal sulphur or hidden spirit," the latter part of which is undoubtedly correct, since:

> Distance (Body) of Mercury = Mercury-Venus Synodic (Soul) = Velocity (Spirit) of Jupiter-Saturn Synodic 4-3.

Moreover, in all three instances and in three separate assignments the parameter for the distance, the period and the velocity is - for the Phi-series - the number 0.381966011 . Which, if rounded at the third decimal place, becomes 0.382 and by multiplying by 1,000 the convenient integer 382 . But wither and whence came the "First of Unities"? And was it something more, or possibly an alternate event after noting Marcilio Ficino's observation that "the chain is unknown, by which Venus and Mercury copulated together in due proportion"? More intriguing is the apparent linkage between the two massive superior planets Jupiter and Saturn with the two innermost and the smallest, with Earth, perhaps errantly, somewhere in between.

On the other hand, after the positive results obtained from the mean value Pheidian and Fibonacci ratios inherent in the motions of these two gas giants, and knowing that for the Phi-series that the inverse velocity for Synodic 4-3 between the two is ${ }^{2}$ with its reciprocal the relative velocity Vr , one might start by using the Fibonacci ratio 13/5 $=2.6$ versus $2.61803398 \ldots$ and take the reciprocal, or simpler yet, use the Fibonacci ratio $5 / 13=0.38461538 \ldots$ Or again use 2.604 to similarly obtain $0.384024 \ldots$ which readily rounds to 0.384 . It is better, however, to concentrate on the relative motions of Jupiter with respect to Saturn which already have some peculiarities in the case of the data for the former. These include an unexpected difference between the ranges for Babylonian Systems $A$ and $B$ for Jupiter with nothing comparable known for Saturn. How might this be linked to the First of Unities? Simply by investigating the relative motions and the two sets of data for Jupiter and Saturn to this end, but only after checks using modern data to provide a standard for their historical equivalents.

Thus for the modern Solar System, using 29.42351935 years $\left(T_{1}\right)$ and 11.85652502 years $\left(T_{3}\right)$ for the mean periods of revolution and applying Phi-series/synodic relation (1), the synodic period or lap cycle of Jupiter with respect to Saturn $\left(S_{2}\right)$ is 19.85887209 years with a velocity $(V r)$ of $0.36927 \ldots$ versus the target value of 0.384 . Similarly, for the Babylonian mean periods (11.86111* and 29.444* years) the synodic period $\left(S_{2}\right)$ is 19.86220818 and the velocity is 0.36925 , thus largely unhelpful results. But this is far from the end of the matter, especially in terms of the Phi-series and Babylonian methodology.

First of all, the data for the periods of revolution and the intervening synodic cycles generated by the Phi-series ( ${ }^{x}, x=-3-2,-1,0,1,2 . .7$ ) involve the use of odd and even exponents for the planets and synodic cycles respectively with exponents -3 and -2 for the periods of Mercury and the synodic cycle between the latter and Venus, whereas exponents 5, 6 and 7 are used for Jupiter, Synodic 4-3 and Saturn. For the Synodic 4-3 data the period is therefore ${ }^{6}$ with the relative velocity $V r$ obtained from $\left({ }^{6}\right)^{-1 / 3}={ }^{-2}=0.381966011$ and the Distance $R$ for Mercury derived from $\left(^{-3}\right)^{2 / 3}={ }^{-2}$. Lastly, the period of Synodic 8-7 remains ${ }^{-2}$ as generated by the Phi-series and incorporated in Tables 3 and 3s. Plus, of course, a more accurate though lower value for the First of Unities of 382.

Secondly, as already seen, the calculation of the mean synodic arc in Babylonian astronomy involves a further period other than the mean periods of revolution and synodic difference periods, namely the parameter $P$ which involves both the period of revolution $P=(T-1)$ and $P=360 /(\mathrm{u})$, the number of mean synodic arcs per revolution of $360^{\circ}$. Therefore, for Jupiter $P_{\text {J }}$ is $10.86111^{*}$ years and for Saturn $P_{s}$ is $28.444^{*}$ years with the synodic period now 17.56994909 and the corresponding velocity 0.38465 ...Furthermore, the same procedure applied to the modern periods likewise results in a velocity of $0.38468 \ldots$ thus purists would likely round to 0.385 , but from a practical viewpoint 0.384 might nevertheless be preferred in both instances.

This is still not quite the end of this "short" excursus, because the two new synodic periods are both close to the Phi-series' ${ }^{6}$ (17.94427191) versus 17.56994909 and 17.56593320 , with 1.612359016 and 1.6122975888 from the sixth roots of the new periods versus Phi itself. These results and others like them helped to provide the impetus for the real-time investigations of planetary motion for these two planets in particular discussed further in Part III.

So much, then for the First of Unities and the minor incursions into the complexities of past and present Alchemy leaving only one thing remaining. Which is the issue arising from the data from Aaboe's 1964 limited research into a possible Babylonian System B augmented here by Friberg's approach to parameters in the ratios of 80/81 and 81/80, thus reciprocals and in the format adopted for the Babylonian velocities, $M=1 ; 00,45^{\circ}$ and $u=0 ; 59,15,33,20^{\circ}$. To which can be added (in terms of symmetry) the minimum arc $m=0 ; 57,46,6,40^{\circ}$.

Next after the conversion from arcs to time and again to distance relative to unity, the ratio of the distances $\left(M_{d}-m_{d}\right) /\left(M_{d}+m_{d}\right)$, i.e., Green's method ${ }^{96}$ for the calculation of eccentricities, yields 0.01677279 versus 0.016708617 from modern estimates. ${ }^{97}$

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[^0]:    And now we might add something concerning a certain most subtle spirit which pervades and lies hid in all gross bodies; by the force and action of which spirit the particles of bodies attract one another at near distances, and cohere, if contiguous; and electric bodies operate to greater distances, as well repelling as attracting the neighboring corpuscles; and light is emitted, reflected, refracted, inflected, and heats bodies; and all sensation is excited, and the members of animal bodies move at the command of the will, namely, by the vibrations of this spirit, mutually propagated along the solid filaments of the nerves, from the outward organs of sense to the brain, and from the brain into the muscles. But these are things that cannot be explained in few words, nor are we furnished with that sufficiency of experiments which is required to an accurate determination and demonstration of the laws by which this electric and elastic spirit operates.

    Sir Isaac Newton, Mathematical Principles of Natural Philosophy.

[^1]:    [in 6,]42 days from (one) appearance to the (next) appearance (Jupiter) moves 30 degrees.' This is correct for the slow arc of System A' for Jupiter. For lines 4 and 5 Sachs suggests: "Compute(?) for the whole zodiac (or for each sign) the appearance according to the day and the velocity.' This is followed in lines 5 and 6 by the rules for the periods: 'in 12 years you add $4 ; 10$, in 1,11 ( 71 years) you subtract 5 , in 7,7 ( 427 years) the longitude (returns) to its (original) longitude'... ' In the Almagest the correction for 71 years is $-4 ; 50^{\circ}$ (IX3, Heiberg p. 215; Manitius II p. 100.

[^2]:    ... Completely in the dark, however, remains the number ...]13,30,27,46 in lines 2 and 3 . In line 3 it seems to be called "mean value". The same number occurs in the parallel passage of No. 813 Section $12 \ldots$ (where) the number [13.30] 27,46 seems to be added to 45,14 which is the mean value of $\Delta \mathrm{T}$."

    In line 4 the difference $\mathrm{d}=1 ; 48$ is multiplied by the number period $\mathrm{II}=6,31$. The result $(11,43 ; 48)$ is not preserved but there is hardly space left for more than the number which would represent the total variation $2(M-m) Z$ of $\Delta \mathrm{T}$. In line 2 the "mean value" (?) . . ] 13,30,27,46 is multiplied by II, again for unknown reasons."

[^3]:    * Table 1, "Velocity Expansions of the Laws of Planetary Motion," JRASC, Vol. 83, No. 3, p. 211, June 1989. Reproduced with permission of the Editor.

[^4]:    Thus this world-machine, consisting of a simple number squared and cubed, leads to a sovereign concord, not only of the ternary to 27 but of the senary to 162 and from 384 to 10,368 , as we have set them out above.

    Francesco Giorgi, Harmonia Mundi (Venice, 1525)

