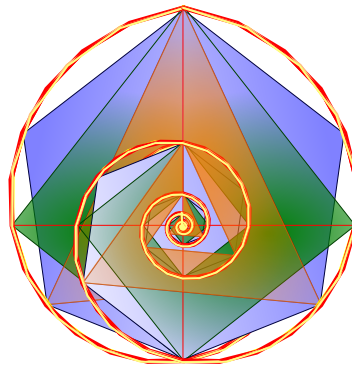


INTIMATIONS OF COMMONALITY IN PLANETARY SYSTEMS



2023

PART ONE

THE PIERCE PLANETARY FRAMEWORK (1850) REVISITED.
plus
EXCURSUS

INTRODUCTION

In the 1850s American scientist Benjamin Peirce (1809–1880) produced a robust heliocentric planetary framework by applying Fibonacci-based reduction ratios to the mean periods of revolution of the eight Solar System planets.¹ Partially incomplete in dynamic terms and subjected to alternative viewpoints, this promising approach was oddly dismissed despite the attendant ramifications and total absence of any comparable planetary theory. Fortunately, however, a condensed version was at least preserved by Louis Agassiz in the latter's *Essay on Classification* (1859).²

As described in the latter work, Peirce began by assigning the outermost planet Neptune a convenient (albeit high) mean period of revolution of 62,000 days. Next, moving inwards, planetary periods rounded to the nearest day were derived from planet-to-planet reduction factors formed from Fibonacci numbers (1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, etc.), specifically, successive *alternate Fibonacci ratios* of 1/2, 1/3, 2/5, 3/8, 5/13, 8/21, 13/34 and 21/55. Thus the 62,000-day period of Neptune was reduced by one-half to obtain a mean period of revolution for Uranus of 31,000 days followed by a one-third reduction of the latter to produce a 10,333-day period for Saturn, 4,133 days for Jupiter (2/5), and so on down to an 87-day period for the innermost planet Mercury from a final reduction ratio of 13/34. However, despite this encouraging end-to-end correspondence a reduction factor for Earth was entirely absent from the alternate Fibonacci sequences. In fact, the inclusion of the latter required two *additional* reduction ratios of 8/13 and 13/21. The last ratio in Pierce's original list (21/55) remained unused but was most likely included for continuity and support for the latter's contention that "There can be no planet planet exterior to Neptune, but there may be one interior to Mercury."³

The Fibonacci-based reduction ratios, resulting periods and comparison with 19th Century Solar System periods were published in the *Essay on Classification* in two sparse, unlabelled tables⁴ based on subdivisions of the 62,000-day period for Neptune. The initial results are shown in Table 1a with title and column assignments added:

PLANETS (ca. 1850)	PERIODS actual/days	PERIODS (reductions)	RATIOS (Pierce)
Neptune,	60,129	62,000	
Uranus,	30,687	31,000	1/2
Saturn,	10,759	10,333	1/3
Jupiter,	4,333	4,133	2/5
Asteriods,	1,200 to 2,000	1,550	3/8
Mars,	687	596	5/13
Earth,	365	366	8/13 } 8/21
Venus,	225	227	13/21 }
Mercury,	88	87	13/34

Table1a. The initial planetary structure, Peirce (1852:129)

Next, the planetary framework was extended to include *twinned ratios* provided by *adjacent* Fibonacci numbers. This produced the same periods of revolution for the planets plus intermediate periods on either side with Earth in an intermediate location between Mars and Venus. Pierce included the intermediate positions for comparable 19th Century data in the fourth column, but apart from 365 days for Earth no other intermediate periods were given. The final ratios and reductions are shown in Table 1b, again with the title and column assignments added:

PLANETS —?—	RATIOS (Pierce)	PERIODS I (reductions)	PERIODS II (actual/days)
Neptune	1/1	62,000	60,129
"	1/1	62,000	
Uranus	1/2	31,000	30,687
"	1/2	15,500	
Saturn	2/3	10,333	10,759
"	2/3	6,889	
Jupiter	3/5	4,133	4,333
"	3/5	2,480	
Asteriods,	5/8	1,550	1,200
"	5/8	968	
Mars	8/13	596	687
Earth	8/13	366	365
Venus	13/21	227	225
"	13/21	140	
Mercury	21/34	87	88

Table1b. The Final planetary structure, Peirce (1852:129)

The final framework languished in this unfinished form despite correlations which included the Mars-Jupiter gap plus the possibility that planet Earth may, perhaps, be occupying an *intermediate* location. This troubling indicator should surely have been investigated, beginning, one might suggest, with mean synodic motion in general and mean synodic lap-cycles in particular.

Mean synodic motion and the intermediate periods

In fact, *all of the intermediate intervals* introduced by Pierce are the *mean synodic periods between adjacent planets*. In other words, lap-cycle times faster-moving inner planets require to complete 360° of direct orbital motion with respect to that of slower-moving outer planets. Adjacent or otherwise, mean synodic periods (S) between planets with mean periods of revolution T_1 and T_3 are derived from the lesser used *general synodic formula*:

$$\text{Synodic period } S_2 = \frac{T_1 \cdot T_3}{T_1 - T_3} \quad (T_1 > S_2 > T_3) \quad (1)$$

although in modern practice relation (1) is rarely applied in this form. Synodic periods in planetary tables normally pertain to either the lap-cycles of *Earth* with respect to the *slower* outer (superior) planets or the lap-cycles of the *faster* inner (inferior) planets with respect to *Earth* itself. In both cases, with the reference period of Earth exactly one year, redundant multiplications by unity are unstated, resulting in the standard synodic formulas:

$$\text{Superior planets } S_s = \frac{T_s}{T_s - 1} \quad (1s) \quad \text{Inferior planets } S_i = \frac{T_i}{1 - T_i} \quad (1i)$$

Nevertheless, relation (1) is more useful in the present context, as is relation (2), where, with both the outer period T_1 and intermediate period $S (= T_2)$ known, the innermost period T_3 can be obtained from:

$$\text{Inner period } T_3 = \frac{T_1 \cdot T_2}{T_1 + T_2} \quad (T_1 > T_2 > T_3) \quad (2)$$

Relation (1) permits the restoration of the missing intermediate periods in Table 1b, and allied with relation (2) plus period formulas employing geometric means - relations (4) and (4E) introduced later - all have roles to play in tests on external planetary systems that follow. More immediately, with missing synodic periods supplied and dynamic component incorporated, a standard planetary framework predicated on Peirce's Fibonacci-based approach can now be assembled as follows.

Units of time and measure

Standard years with respect to *unity* and also the Julian year of 365.25 days are applied in the present study, the first for comparison with modern periods in Julian years,⁵ and the second for real-time calculations of planetary motion in Part Three utilising the methodology developed by Bretagnon and Simon (1986).⁶

Standard order, positions and titles

Following the order adopted by Pierce, the mean periods of revolution and the mean synodic intervals have been assigned standard position numbers and uniform titles commencing with the first and outermost planet. Thus for the eight-planet Solar System the relative *synodic* period (or lap-cycle) of Planet #2 (Uranus) with respect to that of outermost Planet #1 (Neptune) is *Synodic 2-1* followed by *Synodic 3-2* between Planet #3 (Saturn) and Planet #2 (Uranus), and so on, down to *Synodic 8-7* between innermost planet Mercury (#8) and Planet #7 Venus. Planetary positions interior to Mercury (Intra-Mercurial-Objects, or IMOs) commence at IMO 1 followed by IMO 2, etc., with the intermediate synodic periods, Pierce reduction ratios and later divisors continuing inwards in due order. In this theoretical framework, Earth (with reservations) occupies the *Synodic 7-6* location between #6 Mars and #7 Venus.

Divisors for the sequential periods of revolution and intermediate synodic intervals

Next, the awkward multiplications by successive reduction factors used by Pierce are replaced by a standard set of divisors applied to the base period alone, a practice already in use for exoplanets. Thus for the eight-planet Solar System the standard integer divisors for the periods of revolution of the planets beginning with the outermost (#1) are: **1, 2, 6, 15, 40, 104, 273, 714**. Divisors for the intermediate mean synodic periods (lap-cycles) are in turn: 1, 4, 9, 25, 64, 169, 441, thus the synodic divisors are all *sequential squares of the Fibonacci Series*.

The complete set of divisors with intermediate synodic divisors shown in brackets is therefore: **1, (1) 2, (4) 6, (9) 15, (25) 40, (64) 104, (169) 273, (441) 714**, plus (1156) **1870**, (3025) and **4895** for ten-planet systems, etc.

Base periods B1, B2, B3, B4 and B5 for the divisors

Although the period of revolution of the outermost planet (base period B1) is of fundamental importance in Pierce's planetary model, the calculated value for *Synodic* 2-1 is in fact 62,620 days (hereafter base period B2) which exceeds the latter's initial base period of 62,000 days (hereafter, one-off base period P2). Nevertheless, when used as the base period for the divisors, *Synodic* 2-1 yields marginally superior results compared to those obtained with P2. Therefore, where *Synodic* 2-1 differs from B1 a second base period (B2) can be added for further testing. Other bases (B3s) can be approximated by applying the planetary divisors in reverse, *i.e.*, as multipliers of known periods with known locations in otherwise incomplete systems. Where advantageous, the mean value (B4) of multiple B3 products and/or a substitute B5 (yielding least errors) may also be applied at the expense of further complexity.

Resonant triples between planets [RZT]

Resonant triples between planets are included for completeness in Solar System Table 2 and elsewhere. Related to both the twinned Pierce ratios and added divisors, resonant triples are obtained from the bracketing periods of revolution of adjacent planets and the synodic periods in between. Thus, for Neptune and Uranus [1(1)2], Uranus and Saturn [1(2)3], Saturn and Jupiter [2(3)5], etc. Their immediate relevance lies in the fact that the associated divisors are sequential Fibonacci multiples with the central value of each triple providing the multiplication factor. - 1x for the first set: [1(1)2], 2x for the second, thus [2(4)6], 3x for the third [6(9)15], 5x for the fourth [15(25)40], etc.

Fibonacci Periods in days below Mercury

The resulting Pierce P2 planetary framework for a thirteen-planet extension of the Solar System is shown in Table 2a with intermediate positions for the synodic periods and division of modern periods (Base B2/Divisors) included for comparison. The paired resonances from Unity to the Major Sixes (the reverse of the Pierce *reduction* ratios) aid the analyses of exoplanetary systems in Part Two while also bringing to mind ancient methodology, *e.g.*, "Music of the Spheres," which, though not music *per se*, nevertheless appears to have a role in this complex matter. As does the presence of the Fibonacci series below Mercury expressed in days generated by the P2 and the B2 divisors also included in the Table.

PLANETS	N	RATIOS	DIVISORS	RESONANCES	RES.TRIPLES	PERIODS1	PERIODS1 T	DISTANCES1 R	PERIODS2
Synodic	#	(Pierce)	(added)	(to Major 6's) ^a	(to IMO 5)	P2/Divisors	(JYR =365.25)	(Ref. unity/a.u)	B2/Divisors
Neptune	1	1/1	1	1 : 1		62,000	169.74675	30.657329	62,620
<i>Synodic</i> 2-1		1/1	1	1 : 1	1(1)2	62,000	169.74675	30.657329	62,620
Uranus	2	1/2	2	Octave #1, 2 : 1	1(2)3	31,000	84.873374	19.312907	31,310
<i>Synodic</i> 3-2		1/2	4	Octave #2, 4 : 2		15,500	42.436687	12.166369	15,655
Saturn	3	2/3	6	Fifth #1, 6 : 4	2(3)5	10,333	28.291125	9.2846772	10,437
<i>Synodic</i> 4-3		2/3	9	Fifth #2, 9 : 6		6,889	18.860750	7.0855348	6,958
Jupiter	4	3/5	15	Major 6 #1, 15 : 9	3(5)8	4,133	11.316450	5.0404993	4,175
<i>Synodic</i> 5-4		3/5	25	Major 6 #2, 25 : 15		2,480	6.7898700	3.5857029	2,505
M-J Gap	5	5/8	40		5(8)13	1,550	4.2436687	2.6211647	1,566
<i>Synodic</i> 6-5		5/8	64			986	2.6522930	1.9160830	Fibonacci ^b 978
Mars	6	8/13	104		8(13)21	596	1.6480267	1.3952204	610 602
Earth/ <i>Syn</i>	7-6	8/13	169			366	1.0044186	1.0029436	377 371
Venus	7	13/21	273		13(21)34	227	0.6217830	0.7284938	233 229
<i>Synodic</i> 8-7		13/21	441			140	0.3849133	0.5291457	144 142
Mercury	8	21/34	714		21(34)55	87	0.2377405	0.3837681	89 88
<i>Synodic</i> 9-8		(21/34)	1156			54	0.1468397	0.2783315	55 54
IMO 1	9	(34/55)	1870		34(55)89	33	0.0907737	0.2019792	34 33
<i>Synodic</i> 10-9		(34/55)	3025			20	0.0561146	0.1465719	21 21
IMO 2	10	(55/89)	4895		55(89)144	13	0.0346776	0.1063406	13 13
<i>Synodic</i> 11-10		(55/89)	7921			8	0.0214300	0.0771521	8 8
IMO 3	11	(89/144)	12816		89(144)233	5	0.0132449	0.0559800	5 5
<i>Synodic</i> 12-11		(89/144)	20736			3	0.0081861	0.0406179	3 3
IMO 4	12	(144/233)	33552		144(233)377	2	0.0050592	0.0294706	2 2
<i>Synodic</i> 13-12		(144/233)	54289			1	0.0031267	0.0213826	1 1
IMO 5	13	(233/377)	87841			1	0.0019324	0.0155144	1 1

Table2a. The enhanced planetary structure: ratios, divisors, triples, periods in days & years; P2 distances (a.u.).

^a Octave 2 : 1, Fifth 3 : 2, Major Six 5 : 3.

^b The extension to Planet 13 concludes at Fibonacci number 1.

The Solar System revisited

Table 2b shows the uniform assignments, the twinned Pierce ratios, added divisors, resonant triples and the results generated by Pierce base P2, followed by modern base periods B2 and B1 with the latter in both Julian years and days. Also included with the two sets of data are the calculated synodic periods, Mars-Jupiter geometric mean between the periods of the latter pair, associated synodic positions on either side, and Earth located in the synodic position between Venus and Mars. The 365.25 day period for Earth is substituted in the second set of modern data although the actual synodic period (*Synodic* 7-6) is 335 days, thus less than one year and 366-day period obtained from the P2 ratio and 62,000-day base period. Atypical Venus-Earth and Earth-Mars synodic periods and the Mars-Jupiter synodic cycle are omitted for clarity. The periods in days are rounded; red periods equal exact Fibonacci numbers.

PLANETS N Synodics #	RATIOS (Pierce)	DIVISOR (added)	RES.TRIPLE [(RZT)]	PERIODS1 P2/Divisors	PERIODS2 Actual/days	MODERN1 B2/Divisors	MODERN2 B1Julian yrs	MODERN1 (Days)	MODERN2 (Days)
Neptune 1	1/1	1		62,000	60,129	171.44429	163.72320	62,620	59,800
<i>Synodic</i> 2-1	1/1	1	1(1)2	62,000	62,672	171.44429	171.44429	62,620	62,620
Uranus 2	1/2	2		31,000	30,687	85.722145	83.747407	31,310	30,589
<i>Synodic</i> 3-2	1/2	4	1(2)3	15,500	16,658	42.861072	45.360219	15,655	16,568
Saturn 3	2/3	6		10,333	10,759	28.574048	29.423519	10,437	10,747
<i>Synodic</i> 4-3	2/3	9	2(3)5	6,889	7,255	19.049366	19.858872	6,958	7,253
Jupiter 4	3/5	15		4,133	4,333	11.429619	11.856525	4,175	4,331
<i>Synodic</i> 5-4	3/5	25	3(5)8	2,480	2,867	6.8577717	7.8476788	2,505	2,866
M-J Gap 5	5/8	40		1,550	1,725	4.2861072	4.7221497	1,556	1,725
<i>Synodic</i> 6-5	5/8	64	5(8)13	986	1,142	2.6788170	3.1255291	978	1,142
Mars 6	8/13	104		596	687	1.6485028	1.8807111	602	687
Earth/ <i>Syn</i> 7-6	8/13	169	8(13)21	366	335	1.0144633	1.0000000	371	365
Venus 7	13/21	273		227	225	0.6280011	0.6151826	229	225
<i>Synodic</i> 8-7	13/21	441	13(21)34	140	145	0.3887626	0.3958008	142	145
Mercury 8	21/34	714		87	88	0.2401186	0.2408445	88	88
<i>Synodic</i> 9-8	21/34	1,156	21(34)55	54	55	54.169573	54.689759	54	55
IMO 1 9	34/55	1,870		33	34	33.486645	33.723773	33	34
<i>Synodic</i> 10-9	34/55	3,025	34(55)89	20	21	20.700835	20.860438	21	21
IMO 2 10	55/89	4,895		13	13	12.792651	12.888208	13	13
<i>Synodic</i> 11-10	55/89	7,921	55(89)144	8	8	7.9055709	7.9663542	8	8
IMO 3 11	89/144	12,816		5	5	4.8860820	4.9232407	5	5
<i>Synodic</i> 12-11	89/144	20,736	89(144)233	3	3	3.0198701	3.0427860	3	3
IMO 4 12	144/233	33,552		2	2	1.8663570	1.8805320	2	2
<i>Synodic</i> 13-12	144/233	54,289	144(233)377	1	1	1.1534570	1.1622358	1	1
IMO 5 13	233/377	87,841		1	1	0.7128793	0.7183005	1	1

Table 2b. The complete framework and the Solar System. Positions, ratios, divisors and Base periods P2, B1, B2.

Solar System Periods, Pierce Ratios and Divisors below Mercury

Originally the inner region was limited to *Synodic* 9-8 and Planet 9 (IMO 1) to accommodate Pierce's unused inner reduction ratio of 21/55. Accordingly, relation (2) was applied twice, firstly to the mean periods of *Synodic* 8-7 and Mercury resulting in 54.689759 days for *Synodic* 9-8, and then once again to the latter period and that of Mercury to obtain 33.723773 days for Planet #9. However, the last two rounded periods are clearly sequential Fibonacci numbers 55 and 34, an occurrence that allied with the previous sequential pair of periods (145 and 88 days versus Fibonacci 144 and 89) provided the impetus to extend the range as far as Planet 13 (IMO 5) in Tables 2a and 2b.

Regarding the present location of Earth *near* the Mars - Venus *synodic* position, the calculated synodic period, *i.e.*, *Synodic* 7-6 = 335 days represents an enigma since it is neither 366-days as required by the divisors, nor it is close to the actual 365.25 days (Julian) and other variants for the year. Although perhaps masked by a possible outward shift by Mars, this still does little to explain the obvious Fibonacci/*Phi* ratio exhibited by the Venus-Earth periods of revolution expressed in years. In more detail, using modern values for these two adjacent planets the mean periods are 0.61518257 : 1, whereas the reciprocals of *Phi* and Earth (Unity) are 0.61803398875 : 1. Furthermore, there is also the well-known 5 : 8 ratio between the two planets and associated 5 : 8 : 13 *Fibonacci* resonant triple, *i.e.*, 5 synodic periods of Venus in 8 years with 13 corresponding periods of revolution for this planet. All of which, in addition to the above Fibonacci data from Mercury through IMO5, leads logically enough to the following major expansions.

The Pierce planetary framework, the *Phi*-series, and the structure of the Solar System

It is abundantly clear from Table 1b that the final Pierce reduction ratios are successive twinned members of the Fibonacci series, albeit one position removed between the numerators and denominators. Nevertheless, despite the title of Pierce's original publication¹ and obvious nature of the ratios applied by the latter, the Fibonacci series and related Golden Ratio *Phi* (ϕ) :

$$\text{Phi}(\phi) = \sqrt{(5/4) + 1/2} = 1.618033988749895 \quad (3)$$

– are nowhere stressed by Peirce or Agassiz, although this constant clearly plays a major role in the proposed model. This is all the more apparent when it is recalled that the golden *section* is defined as the division of a line such that the proportion of the smaller section to the larger is identical to the proportion of the larger section to the whole. Whereas the golden *ratio* can be defined as the limiting value of the ratios of adjacent Fibonacci numbers. It is also clear in the present astronomical context that moving inwards, the limiting value of the inverse alternate Fibonacci ratios applied by Peirce will be ϕ^{-2} (0.38196601125) with reciprocal limit the outward multiplier ϕ^2 (2.61803398875). Furthermore, after the inclusion of the ratios for the intermediate periods between planets the limiting value is ϕ^{-1} (0.61803398875) with a reciprocal limit and a corresponding multiplier of ϕ^1 (1.61803398875), which is *Phi* itself.

The *Phi*-series in astronomical context (*Periods T, S years, Distance R, Velocity Vi and Vr relative to unity*)

As it so happens, apart from filling the intermediate gaps introduced by Pierce, relation (1) – the general synodic formula – is already present with one central exception among the four constants just mentioned, i.e., ϕ^{-2} , ϕ^{-1} , ϕ^1 and ϕ^2 . In short, combined with the calibration and the unification provided by the mean period of Earth ($\phi^0 = 1$ year) the latter become sequential mean periods *in years* generated by the *Phi*-series ϕ^x for successive integer exponents $x = -2, -1, 0, 1, 2$ in the present context. Moreover, with the addition of the next lower integer and also continued outward extensions, integer exponents -3 through 7 generate a complete planetary framework from Mercury to Saturn with all synodic periods included. Beyond this outer region correlation with the solar system parameters begins to diminish, but nevertheless, for the stipulated range the inter-related parameters are as shown in Table 3. Here, following ancient practice it is helpful to include the *inverse velocity* V_i , e.g., $V_i^2 = R$, $V_i^3 = T$, $V_i^{-1} = V_r$ (best remembered by the Triple interval $[3^0 3^1 3^2 3^3 = 1, 3, 9, 27]^7$ which also pertains to Saturn at perihelion) with the frame of reference (unity) provided by the mean heliocentric distance (R) in a.u, mean period of revolution (T) in years and mean orbital velocity (V_r) of Earth. Thus in the same sense $[1, 1, 1, 1]$, hence the assignment of the cube to planet Earth and *tetradic* point-line-area-volume analogy applied to planetary motion. The last three modern periods in days (in red) owe their origins to relation (2) and a 33.0225-day period for IMO 1 by Leverrier (1875).⁸

PLANETS N Synodics #	MODERN T (Julian Years)	$\phi^x = \text{Phi-series } T$ x	$\text{Phi-series } T$ (Years)	<i>Phi</i> -series (R) Distance (a.u.)	<i>Phi</i> -series (Vi) Inverse Velocity	<i>Phi</i> -series (Vr) Velocity (Ref.1)	MODERN T (Days)
Saturn 3	29.4235194	7	29.03444185	9.446602789	3.073532624	0.325358512	10746.9404
Synodic 4-3	19.8588721	6	17.94427191	6.854101966	2.618033989	0.381966011	7253.45303
Jupiter 4	11.8565250	5	11.09016994	4.973080251	2.230040414	0.448422366	4330.59576
Synodic 5-4	7.84767877	4	6.854101966	3.608281187	1.899547627	0.526441130	2866.36470
M-J Gap 5	4.72214968	3	4.236067977	2.618033989	1.618033989	0.618033989	1724.76517
Synodic 6-5	3.12552908	2	2.618033989	1.899547627	1.378240772	0.725562630	1141.59949
Mars 6	1.88071105	1	1.618033989	1.378240772	1.173984997	0.851799642	686.929711
Earth/Syn 7-6	0.91422728	0	1.000000000	1.000000000	1.000000000	1.000000000	365.25(JYR)
Venus 7	0.61518257	-1	0.618033989	0.725562630	0.851799642	1.173984997	224.695433
Synodic 8-7	0.39580075	-2	0.381966011	0.526441130	0.725562630	1.378240772	144.566223
Mercury 8	0.24084445	-3	0.236067978	0.381966011	0.618033989	1.618033989	87.9684354
Synodic 9-8	0.14474748	-4	0.145898034	0.277140264	0.526441130	1.899547626	54.6897591
IMO 1 9	0.09041068	-5	0.076806725	0.201082619	0.448422366	2.230040414	33.0225000
Synodic 10-9	(0.0556507)	-6	0.055728090	0.145898034	0.381966011	2.618033989	20.3264209
IMO2 10	(0.0344447)	-7	0.040434219	0.105858161	0.325358512	3.073532624	12.5818709

Table 3. Modern periods T , S , *Phi*-series, exponents (x), T , R , Velocity V_i (Inverse) and V_r (relative to unity).

Returning to the present, notwithstanding the Mars-Jupiter Gap and anomalous location of Earth *between* Mars and Venus, the *Phi*-series planetary framework outlined above includes the following properties and relations:

Heliocentric properties of the *Phi*-series with respect to unity in the Solar System

1. For any three successive *Phi*-series periods, the middle period is the *product* of the periods on either side *divided by their difference*. Thus, in the same astronomical context, the general synodic formula, relation (1)
2. If two upper adjacent *Phi*-series periods are known, the third and lower period can be obtained from the *product of the two adjacent periods divided by their sum*. Thus (in addition to relation 1), synodic relation (2)
3. The underlying constant of the *Phi*-series planetary model is ***Phi*** ($\phi^1 = \frac{1}{2} \sqrt{5} + \frac{1}{2} = 1.618033988749895$), the limiting value of successive ratios of the Fibonacci series: 1,1,2,3,5,8,13,21,34,55,89,144,233,377, 610,... (3)
4. For any three successive *Phi*-series periods, the middle period is the *geometric mean* of the two periods on either side, as are the means from positions $\pm 2, \pm 3$, etc. Extended geometric means, relations (4), (4E) & (4F)
5. For every *Phi*-series period *except that of Earth* there exists a corresponding Lucas series integer period ($\sim 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, \dots$ years) generated by the *alternating Phi*-Lucas relation: $(T, S) = |\phi^x \pm \phi^{-x}|$
Periods of revolution: $T = |\phi^x - \phi^{-x}|$. Intermediate Synodic Periods: $S = |\phi^x + \phi^{-x}|$ *Phi*-Lucas relation (5)
6. Pertaining to *planet EARTH*, the product of the parameters of the planets on either side (Mars and Venus) is *UNITY*, as are *all such Phi-series products*, i.e., periods $\pm 2, \pm 3$, etc., both inwards and outwards. Relation (6E)

The limiting *Phi*-series constants in the present astronomical context are:

- A: PLANETS: Mean sidereal periods of revolution, mean heliocentric distances, mean orbital velocities:
Phi-series mean *periods* of revolution (*T*) decrease ϕ^{-2} (0.38196601125), Inwards (the Pierce limit) (7)
Phi-series mean *periods* of revolution (*T*) increase ϕ^2 (2.61803398875), Outwards (8)
Phi-series mean heliocentric *distance* (*R*) increase $\phi^{4/3}$ (1.89954762695), Planets, Outwards (9)
Phi-series *Planet-to-Planet Velocities* (*Vr*) decrease $\phi^{-2/3}$ (0.725562630246), Planets, Outwards (10)
- B: SYNODICS: Mean synodic periods, corresponding heliocentric “distances,” mean “orbital” velocities:
Phi-series mean *synodic* (*S*) to *Planet* (*T*) decrease ϕ^{-1} (0.61803398875), Inwards (11)
Phi-series mean *synodic* (*S*) to *Planet* (*T*) increase ϕ^1 (1.61803398875), Outwards (12)
Phi-series mean *synodic* (*R*) to *Planet* (*R*) increase $\phi^{2/3}$ (1.37824077249), Outwards (13)
Phi-series mean *synodic* (*Vi*) to *Planet* (*Vi*) increase $\phi^{1/3}$ (1.17398499671), Outwards (14)
Phi-series mean *synodic* (*Vr*) to *Planet* (*Vr*) decrease $\phi^{-1/3}$ (0.85179964208), Outwards (15)
- C: GENERATION:
The *mean periods* of revolution (*T*) and the *mean synodic periods* (*S*) in years from Mercury to Neptune are generated by the *Phi*-Series (ϕ^x) and integer exponents $x = -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$ and 11. (16)
The *mean periods of revolution* are generated by *ODD* exponents, *mean synodic periods* by the *EVEN*. (17)
- D: OVERALL PLANETARY FRAMEWORK with increasing departures beyond Saturn (periods *T*, *S* in years).
Period divisors, modern values, exponents, Lucas series and *Phi*-series framework are shown in Table 3s.
n.b., the *Phi*-series also includes each key *Pheidian* constant as a mean period (*T*, *S*), a mean distance (*R*) and both velocities (*Vr* & *Vi*) with the latter (0.381966011) at *Synodic* 10-9) not shown. (18)

DIV. (syn)	PLANETS N Synodics #	MODERN T,S (Julian years)	ϕ^x exp.	LUCAS (years)	<i>Phi</i> -series T,S (ϕ^x = years)	<i>Phi</i> -series (R) Distance (a.u.)	<i>Phi</i> -series (Vr) Velocity (ref.1)	<i>Phi</i> -series (Vi) Velocity (Inv.)
1	Neptune 1	163.7232045	11	199	199.0050294	34.08599912	0.171282103	5.838321602
(1)	<i>Synodic</i> 2-1	171.4442895	10	123	122.9918694	24.73152718	0.201082619	4.973080251
2	Uranus 2	83.7474068	9	76	76.01315562	17.94427191	0.236067977	4.236067978
(4)	<i>Synodic</i> 3-2	45.3598213	8	47	46.97871376	13.01969312	0.277140264	3.608281187
6	Saturn 3	29.4235194	7	29	29.03444185	9.446602789	0.325358512	3.073532624
(9)	<i>Synodic</i> 4-3	19.8588721	6	18	17.94427191	6.854101966	0.381966011	2.618033989
15	Jupiter 4	11.8565250	5	11	11.09016994	4.973080251	0.448422366	2.230040414
(25)	<i>Synodic</i> 5-4	7.84767877	4	7	6.854101966	3.608281187	0.526441130	1.899547627
40	M-J Gap 5	4.72214968	3	4	4.236067977	2.618033989	0.618033989	1.618033989
(64)	<i>Synodic</i> 6-5	3.12552908	2	3	2.618033989	1.899547627	0.725562630	1.378240772
104	Mars 6	1.88071105	1	(2)	1.618033989	1.378240772	0.851799642	1.173984997
(169)	Earth/Syn 7	0.91422728	0	1	1.000000000	1.000000000	1.000000000	1.000000000
273	Venus 7-6	0.61518257	-1	-	0.618033989	0.725562630	1.173984997	0.851799642
(441)	<i>Synodic</i> 8-7	0.39580075	-2	-	0.381966011	0.526441130	1.378240772	0.725562630
714	Mercury 8	0.24084445	-3	(Rel. 5)	0.236067978	0.381966011	1.618033989	0.618033989

Table 3s. Divisors, modern periods (*T* & *S*), *Phi*-series (ϕ^x) *T*, *S*, Lucas *T*, *S*, *Phi*-series *R*, Velocity *Vr* & Inverse *Vi*.

Disparities in the modern Solar System from Mercury through Saturn with emphasis on the Pierce Divisors and insights from the above are shown in Figure 1.

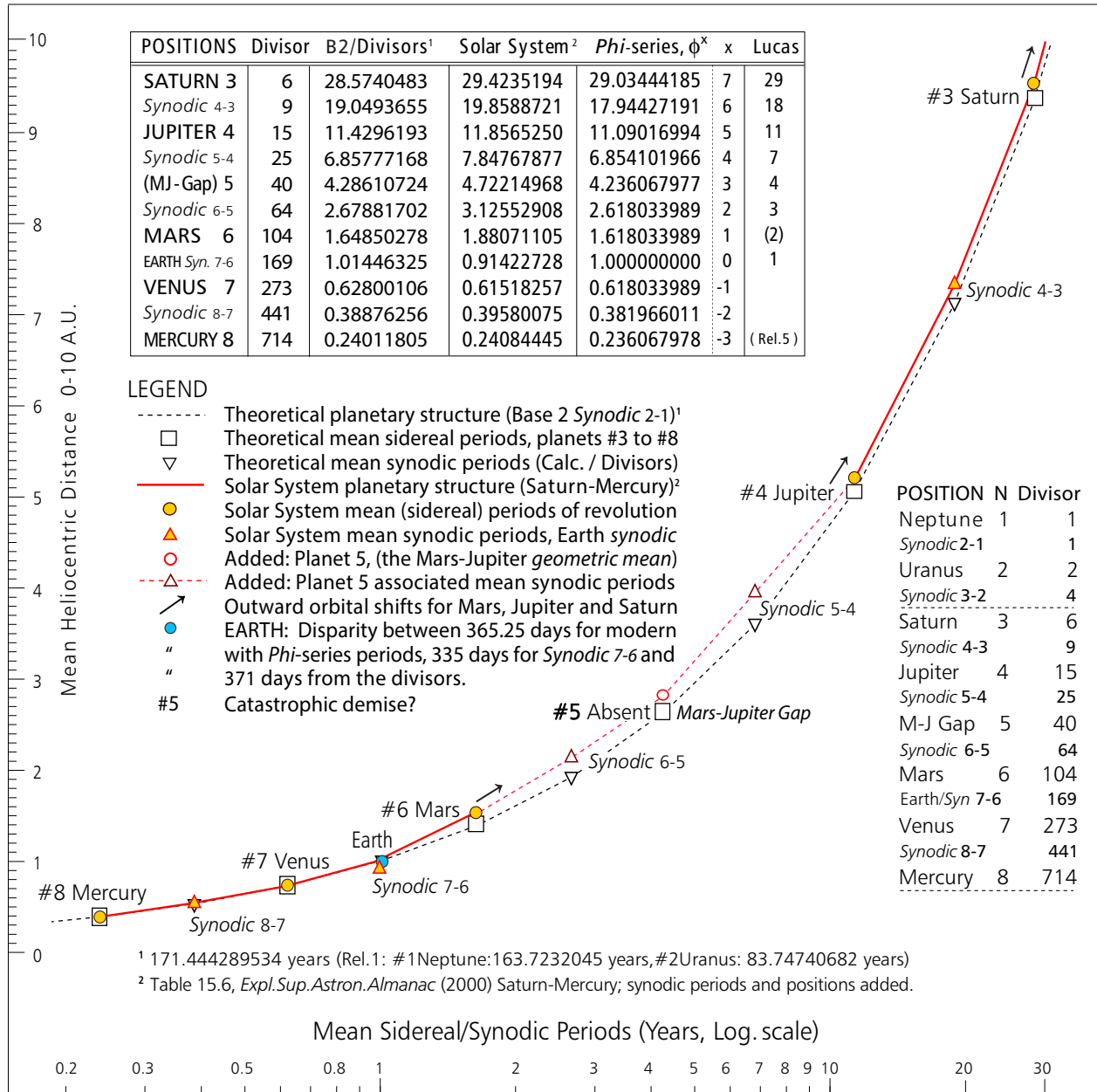


Fig. 1. The Pierce planetary framework, Solar System Mercury–Saturn, Mars-Jupiter Gap and the location of Earth.

Similarities and Disparities

Applied to the present Solar System, the Φ -series from $x = -3$ to 7 yields a planetary framework which includes all the intermediate (synodic) periods from Mercury through Saturn plus periods for theoretical planet #5 and both adjacent intermediate synodic intervals. Beyond Saturn correlation diminishes with distance, while the ratios of the of the integral Lucas series increasingly approach Φ itself. Whereas, moving inwards, ratios of the *period divisors* also begin to approach the same fundamental constant. Nevertheless, two identical disparities in the Solar System are indicated by all three sequences: (1) the absence of a planet between Jupiter and Mars, and (2), the unexpected presence of a planet between adjacent Mars and Venus, namely Earth. Moreover, in addition to this location there is also a marked difference between the calculated intermediate period for Earth of 335 days and the 365-day year.

In so much as Venus and Earth have the lowest eccentricities among the planets and their periods of revolution are also closest to their Φ -series equivalents - with zero error for Earth - the position of the latter can be examined in terms of residual effects of the Φ -series with relation 6E a possible factor. This, however, is difficult to investigate because of the missing periods between Mars and Jupiter, and also the accepted absence of planets below Mercury.

About the only option remaining pertains to the periods of Jupiter and IMO 1, *i.e.*, the periods corresponding to *Phi*-series exponents +5 and -5 which yield a *product* of exactly 1 year. Whereas in the Solar System the mean period of Jupiter of 11.85652502 Julian years and that of IMO 1 (0.09035592 years) yields a product of 1.071307238, with the replacement of IMO 1 by the mean synodic month resulting in 0.9586044 years. *Unity* does, however result from a period of 30.8058220 days from the *reciprocal* of Jupiter's mean period, a concept which owes its origins to Friberg's approach to AO 6484, a Babylonian mathematical text concerned with the number 0;59,15,33,20 and its reciprocal 1; 00,45.⁹ The product is necessarily *unity* with a *sum* of 2; 0,0,33,20 and 1; 0,0,16,40 for the half.⁹ Which, albeit radical shifts in both time and place, can be considered in terms of *elliptical parameters for the orbit of Earth*. This is an unexpected bi-product of a reappraisal of the 1964 analyses by A. Aaboe¹⁰ of a possible *daily increment* of 0; 0,1,32,42,13,20° (0.000480109739369) for the velocity of the "Sun" in BM 37089, a Babylonian lunar fragment. The relevance of the latter is that the value 0;59,15,33,20^{O/Day} can be shown to be inherent in data in Aaboe's study which corresponds to a period of exactly 364.5 years. This value is shown below in the last column of Table 1A from an expansion of Aaboe's analyses incorporating a Babylonian System B *varying velocity* function for planet *Earth*:

"... Although Aaboe surmises that the original table may have supplied daily longitudes for a complete year, he gives a partial restoration since neither the maximum nor the minimum values are present. It is, however, sufficient to give Aaboe's daily longitudes and differences for lines 5 through - 5 plus added corresponding lengths of the year in days to show that line 0 is the closest to the Sidereal year:

Line #	Col. II (Longitude°)	Δ Col. II (Daily velocity°)	T (years, added)
Line -5	[8;51,51,51,6,40]	0;59,17,17,2,13,20	364.32289859
Line -4	[9;51,6,40]	0;59,15,33,20	364.5
Line -3	[10;50,20,29,37,46,40]	0;59,13,49,37,46,40	364.67727367
Line -2	[11;49,32,35,33,20]	0;59,12,5,55,33,20	364.85471987
Line -1	[12;48,42,57,46,40]	0;59,10,22,13,20	365.03233883
Line 0	[13;47,51,36,17,46,40]	0;59,8,38,31,6,40	365.21013081
Line 1	[14;46,58,[31,6,40]	0;59,6,54,48,53,20	365.38809607
Line 2	15;46,[3, 42,13,20]	0;59,5,11,6,40	365.56623485
Line 3	16;45,7,[9,37,46,40]	0;59,3,27,24,26,40	365.74454742
Line 4	17;44,8,[53,20]	0;59,1,43,42,13,20	365.92303402
Line 5	18;43,[8,53,20]	0;59.	366.10169492

Table 1A. Daily solar positions and velocities with periods *T* added to Aaboe (1964:32).

This demonstrates that from a modern perspective the mean daily velocity from line 0 of 0; 59, 8,38,31, 6,40^{O/d} and the 365.21013081-day year are optimum for (*u*) and (*T*) respectively. But not quite. In order to restore the longitudes and velocities for the entire table, the period *T* turns out to be exactly 364 days. Thereafter, with (*d*) given, (*u*) from Line 0, and *T* = 364 days, relation (X) is reduced to: (*M*, *m*) = 0;59,8,38,31,6,40 ± 0;2,37,17,2,13,20 which produces the following six-sexagesimal place values for the apsidal velocities and the daily velocities in between.

Minimum daily velocity (*m*) = 0;56,31,21,28,53,20^{O/day} (abbrev. 0;56,30)
Mean daily velocity (*u*) = 0;59, 8,38,31, 6, 40^{O/day} (abbrev. 0;59, 9)
Maximum daily velocity (*M*) = 1; 1,45,55,33,20,00^{O/day} (abbrev. 1; 1,46)
eccentricity (*e*) = 0.0295589.

The occurrence of 0;59° in line 5 of column 3 suggests choice rather than coincidence and there are other matters of interest in addition." [Excerpt from "Aaboe64 Revisited"].

The above dialogue concludes with an associated ellipse and additional variants which are beyond the scope of the present study. Except to note that Friberg's analysis mentioned earlier is accompanied by a two-part figure for the Babylonian mathematical procedure known as "Completing the Square." The latter, however, in consort with the calculation of the heliocentric distances *R* (by a procedure provisionally named here "Completing the Cube" inherent in Old Babylonian mathematical text VAT 8547) suggests that these procedures ultimately concern the derivation of the parameters of *ellipses* for Earth and the major superior *Planets*. In so much as the eccentricities (*e*) are small (*e.g.*, that of Earth is 0.01670862) the orbits appear to be *almost* circular, which provides an impetus to revisit Babylonian mathematical texts with accompanying "circles" and non-integer numerical values *close to unity* or 2. Thus possible semi-major (*a*) and major axes (2*a*) for Earth/*Sun ellipses*, *e.g.*, although conceivably with alternate meanings:

¹¹"Fig. 3. 1. 12. MS 3050. An OB round hand tablet with square inscribed in a circle." Friberg (2005:135).

¹²"Fig. 16. 7. 3. UET/67 2 222 rev. A square side algorithm using elimination of square factors." Friberg (2006:401)

¹³"Fig. 16. 7. 4. 1st. Si. 428. Computation of the square side 2;02,02,02,05,05,04." Friberg (2006 :403).

where the first figure appears to be a rough *rectangle* with diagonals inscribed in an equally rough *ellipse*.

Seeking further enlightenment the inquiry leads to Babylonian planetary and luni-solar parameters, but before this it is necessary to caution the casual reader about prevailing nihilistic views concerning Babylonian astronomy, especially ill-founded claims that the Babylonians had neither a fictive approach to orbital motion nor any planetary model whatsoever. Long overdue additional research shows that nothing could be further from the truth.

But before proceeding, the notation, conventions and additional data in this context are introduced for those who may be unfamiliar with this relatively obscure material, along with standard definitions of astronomical terms, and in particular, luni-solar and planetary parameters in both modern and Babylonian contexts.

Sexagesimal notation, Units, Time, and Motion

Sexagesimal numbers 1 to 59 are separated by commas with equivalent decimal place locations indicated by semi-colons, thus in addition to hours; minutes and seconds, the thirds, fourths, fifths, sixths, sevenths, etc. For example, the Old Babylonian estimate for the square root of 2 rounded at the *third* place is 1;24,51,10¹³ with the *exact* value for the Babylonian mean synodic arc of Saturn¹³ 12;39,22,30° (12.65625°) with a corresponding mean synodic time of 1;2,6,33,45 years (1.03515625 versus the modern mean synodic period for this planet of 1.035182135... years). Days, degrees, months and “tithis” (thirtieths) are denoted by the superscripts n^d , n° , n^m , n^r with the predominant Babylonian mean synodic month (MSM) of 29;31,50,8,20^d (29.5305941358...^d) represented by superscript m .

Next, expanded later, definitions and tools for the present study include the following luni-solar constants:

- (1) DAY: *Daily axial rotation and daily sidereal motion of Earth* with subdivisions of the 24-hour day for time & angular motion which far exceed modern usage, extending from 360° per day through Large Hours (30°), Hours (15°), Minutes and Seconds, etc., down to 50 seconds of arc (0;00,50°).
- (2) MONTH: MEAN SYNODIC MONTH of 29;31,50,8,20 days = 29.5305941358^d with last base-60 pair rounded for convenience. Even so it is still quite accurate; the modern estimate is 29;31,50,7,30 days.
- (3) YEAR: SIDEREAL YEAR of 12;22,8 Mean synodic months = 365;15,38,17,44,26,40 days (365.2606376886). Although the latter is high compared to the modern estimate of 365.2564^d it is almost certainly selected for convenience. A better estimate for the sidereal year is also available from the accurate Babylonian mean sidereal month of 27;19,18^d and above mean synodic month which generate a year of 365.2564698 days.
- (4) METHODOLOGY: Explanations of the fundamental motions involved according to the methods laid out in the Babylonian procedure texts and related data determined from the Babylonian end products, *i.e.*, the Ephemerides. And in addition, the implications of the Earth/Sun duality in the Babylonian context.
- (5) Closely associated to (4), the underlying formulas required to assess Babylonian results and procedures. In this case, since Babylonian planetary theory deals to a considerable extent with *synodic* motion, and the latter understanding is also applicable to the lunar component, the computation of synodic cycles, synodic periods and synodic arcs also play a role in the current investigation, the following especially:

Synodic periods and synodic formulas

The synodic period (S) or lap-cycle between two Solar System planets with mean periods of revolution T_1 and T_2 is given by the *general synodic formula for co-orbital bodies* applied earlier to the Pierce data :

$$\text{Synodic period } S = \frac{T_1 \cdot T_2}{T_1 - T_2} \quad (T_1 > T_2) \quad (1)$$

along with the simplified standard synodic formulas for the Superior and Inferior planets:

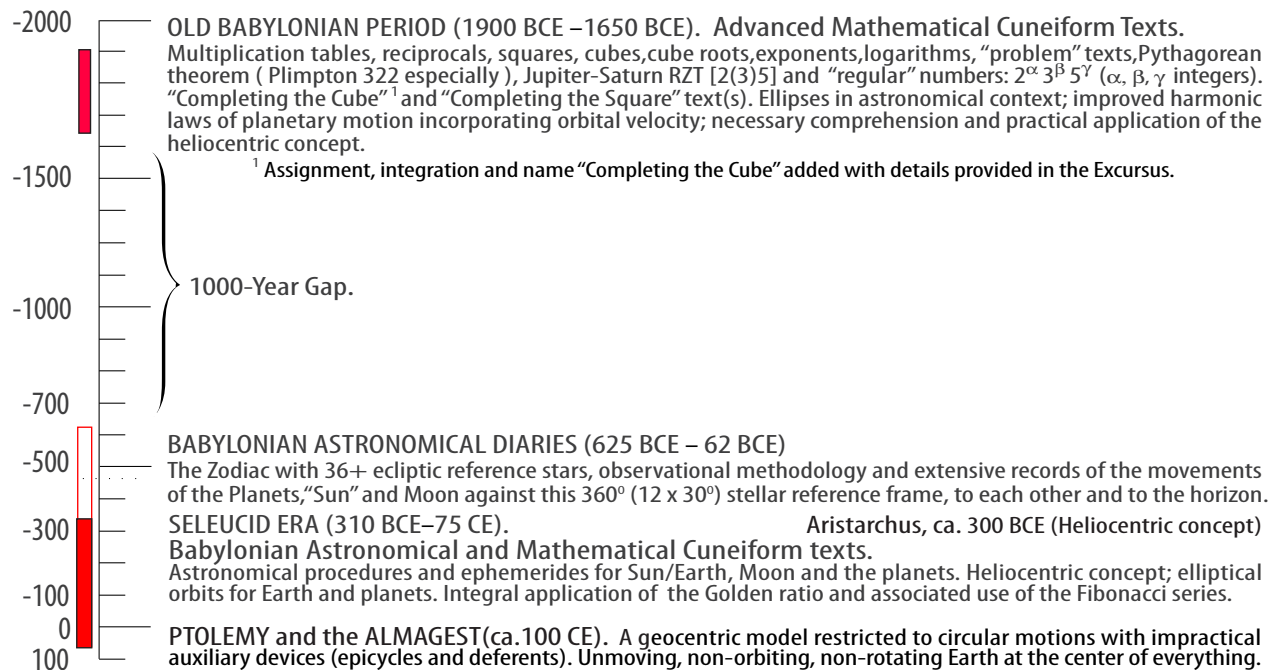
$$\text{Superior planets, } S_s = \frac{T_s}{T_s - I} \quad (1s) \quad \text{Inferior planets, } S_i = \frac{T_i}{I - T_i} \quad (1i)$$

augmented, if required, by synodic relation (2) where periods T_1 and S are known and period T_2 is of interest:

$$\text{Period } T_2 = \frac{S \cdot T_1}{S + T_1} \quad (S > T_2) \quad (2)$$

Synodic relations (1), (2) and modern equivalents all have roles to play in what follows, but relation (1) in full has a further application arising from the inclusion of the mean synodic month in Babylonian planetary theory beyond calendaric considerations. Although obvious, this was either missed or - for whatever reasons - ignored by noted authority Otto Neugebauer. More on this matter later.

As for the relevance of Babylonian astronomy in the presence context, further examination the mathematical cuneiform texts from the Old Babylonian Period (1900 BCE - 1650 BCE),¹⁴ the Babylonian astronomical diaries from 652–62 BCE,¹⁵ details in the Babylonian astronomical “procedure” texts and the resulting *Ephemerides* of the Seleucid Era (310 BCE–75 CE)¹⁶ represent an extensive source of largely misrepresented and/or misunderstood information. Included here are specific parameters with descriptions in the procedure texts concerning their determination, sufficient details, in fact, for the *heliocentric concept* and *refined laws of planetary motion* to be added to the already complex mathematics of the Old Babylonian Period. The acceptance of which is adversely influenced by the time line between the sources and lack of connectivity with the earliest in terms of known astronomical concepts.



Text-Fig 1. Old Babylonian Mathematics, Babylonian Astronomical Diaries and Seleucid Era Astronomy.

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Introduction

Although the roles of synodic relations (1) and (2) with respect to the Solar System periods and synodic cycles are not entirely surprising the two relations are nevertheless both underlying elements of the structure of the *Phi*-series planetary framework. Just how well the Babylonian luni-solar material reflects this is another matter, but assuredly the subject is worthy of further investigation, especially with respect to attested Babylonian periods and velocities for the five planets known in Antiquity. But then again, the *Fibonacci*, *Lucas* and the *Phi*-series are all considered to be relatively recent in both origins and understanding, hence the following introduction to the historical side of the matter.

I. The Fibonacci and the Lucas series in early times.

Although the first of these two elementary series is still credited to Fibonacci (*ca.*1175–1240 CE) and likewise the second to Francois Lucas (1842–1891), as Thompson pointed out over a century ago,¹⁷ it is unlikely that the former would have escaped the attention of Greek philosophers or even earlier inquiring minds. Furthermore, this same argument applies equally (if not more so) to the latter series (1,3,4,7,11,18,29,47) since it is simply the *next additive sequence* after the *Fibonacci*, *i.e.*, 1,1,2,3,5,... is followed by: 1,3,4,7,... (the *Lucas*), then: 1,4,5,9,... and 1,5,6,11,... etc., all with the same limiting ratio (ϕ) between adjacent pairs. The last mentioned (provisionally the *Penta* series 1,5,6,11,17,28,...) also includes the first two perfect numbers 6 and 28 (numbers equal to the sum of their own parts). And eventually, the convenient approximation for the Golden ratio of $809/500 = 1618/1000$, thus **1.618** (1;37,4,48).

II. Babylonian Jupiter/Saturn mean synodic arcs; the *Phi*-series and the Golden ratio

Both historically and in astronomical terms, the ratio **5 : 6** is known to play an underlying role in the location of the extremal synodic arcs for Jupiter¹⁸ and Saturn¹⁹ in the Babylonian astronomical cuneiform texts of the Seleucid Era (310 BCE–75 CE). Furthermore, despite current dismissive views on this subject, another point of relevance is found in the Babylonian estimates for the *sidereal* periods of revolution for Jupiter (11;51,40 = 11.86111* years) and Saturn (29;26,40 = 29.444* years) which provide the basis for the mean synodic arcs (u) according to Babylonian System B. In particular, it is *the ratios* of these synodic arcs - 33;8,45° (33.1458333*) for Jupiter²⁰ and 12;39,22,30° (12.65625) for Saturn²¹ - which are of immediate interest, since:

$$\frac{\text{Saturn } (u)}{\text{Jupiter } (u)} = \frac{12.65625}{33.1458333*} = 0.38183534 \text{ versus } \phi^{-2} = 0.38196601125, \text{ the Pierce Limit, } \textit{Phi}\text{-series relation} \quad (7)_{S/J}$$

$$\frac{\text{Jupiter } (u)}{\text{Saturn } (u)} = \frac{33.1458333*}{12.65625} = 2.61893004 \text{ versus } \phi^2 = 2.61803398875, \text{ Planet-to-Planet } \textit{Phi}\text{-series relation} \quad (8)_{J/S}$$

whereas the difference between the two mean synodic arcs, *i.e.*, Jupiter (u) - Saturn (u) = 20;29,22,30° (20.48958333*) not only provides the arc for the difference cycle *SD1* between the two planets (Synodic 4-3 in the Peirce framework) but also two further inter-related ratios of similar interest:

$$\frac{\text{Jupiter } (u)}{\text{SD1}(u)} = \frac{33.1458333*}{20.48958333*} = 1.61769192 \text{ versus } \phi = 1.61803398875, \text{ Planet-to-Synodic } \textit{Phi}\text{-series relation} \quad (12)_{J/D}$$

(1.61764706 = Fibonacci 55/34)

$$\frac{\text{SD1}(u)}{\text{Saturn } (u)} = \frac{20.48958333*}{12.65625} = 1.61893004 \text{ versus } \phi = 1.61803398875, \text{ Synodic-to-Planet } \textit{Phi}\text{-series relation} \quad (12)_{D/S}$$

III. Babylonian Jupiter/Saturn mean synodic arcs and the Fibonacci series

In addition to this pair of mean synodic arcs, System A' for Jupiter²² features an intermediate arc of 33;45° (33.75° = u_2) as opposed to (u), the mean synodic arc of 33;8,45°. Retaining Saturn's mean synodic arc of 12;39,22,30° but using 33.75° for Jupiter and new difference arc $\text{SD1}' = 21;5,37,30°$ (21.09375) the divisions for the new arcs now yield the following familiar Fibonacci ratios which suggests the previous relations are *unlikely to be coincidental or unknown*;

$$\frac{\text{Jupiter } (u_2)}{\text{SD1}(u)'} = \frac{33.75}{21.09375} = 1.6 \text{ (1;36). Fibonacci ratio (8/5)} \quad (12)_{J/DF}$$

$$\frac{\text{SD1}(u)'}{\text{Saturn } (u)} = \frac{21.09375}{12.65625} = 1.666* \text{ (1;40). Fibonacci ratio (5/3)} \quad (12)_{D/SF}$$

$$\frac{\text{Jupiter } (u_2)}{\text{Saturn}(u)} = \frac{33.75}{12.65625} = 2.666* \text{ (2;40). Fibonacci ratio (8/3)} \quad (8)_{J/SF}$$

IV. Jupiter and Saturn mean value ratios for Babylonian Systems A and B

Remaining with Jupiter and Saturn, there is a major difference between the two primary methods for dealing with *varying* synodic motion (Systems A and B) with the two-velocity configurations of System A using a minimum arc (w) and a maximum arc (W) sensibly understood to be *apsidal* velocities with *pheidian* elements in an associated **5:6** ratio. For Jupiter the minimum and maximum synodic velocities (or apsidal arcs) are 30° and 36° ,²³ whereas for Saturn the minimum (w) and the maximum (W) have a marked difference in the number of sexagesimal places, i.e., (w) = 11;43,7,30° (11.71875), and (W) = 14;3,45° (14.0625).²⁴

On further examination, however, it seems possible that the latter set may have originated from the former since the seemingly more accurate apsidal synodic arcs for Saturn can be derived from the Jupiter data by *simple division*, i.e., $36^\circ/2.56 = 14;3,45^\circ$ (14.0625) and $30^\circ/2.56 = 11;43,7,30^\circ$ (11.71875). Plus one further point; the common divisor is also the *square* of Fibonacci ratio 8/5 ($1.6^2 = 2.56$), thus a practical reduction factor for the periods of revolution of these two adjacent major planets in keeping with the *Fifths* and the *Sixths* of the final Pierce framework.

It is, however, more complicated than this, for even though Jupiter's new value from System A' ($u_2 = 33;45^\circ$ as used in relations (12)_{J/DF} though (8)_{J/SF}) resulted in three Fibonacci ratios using this constant, it is not the actual mean value for Jupiter, which is $\frac{1}{2}(W + w) = (u_3) = 33^\circ$. Whereas the mean value from Saturn's System A is in turn found to be $\frac{1}{2}(14;3,45^\circ + 11;43,7,30^\circ) = (u_4) = 12;53,26,15^\circ$ (12.890625) with the ratio between the two new mean values now:

$$\frac{\text{Jupiter } (u_3)}{\text{Saturn } (u_4)} = \frac{33}{12.890625} = 2.56 = 1.6^2. \text{ Fibonacci ratio } (8/5)^2 \quad (8)_{J/SF2}$$

while the ratio between Jupiter (u_2) and Saturn (u_4) is:

$$\frac{\text{Jupiter } (u_2)}{\text{Saturn } (u_4)} = \frac{33.75}{12.890625} = 2.6181818^* = \text{Fibonacci ratio } 144/55 \quad (8)_{J/SF3}$$

with a reciprocal of: $\frac{\text{Saturn } (u_4)}{\text{Jupiter } (u_2)} = \frac{12.890625}{33.75} = 0.3819444^* = \text{Fibonacci ratio } 55/144 \quad (7)_{S/JF2}$

and a corresponding ideal growth angle ($360^\circ \cdot 0.3819444^*$) of 137.5° .

Lastly, with a new difference arc $SD1$ of $(33.75 - 12.890625) = 20.859375$ (u)", relation (12)_{J/DF} now becomes:

$$\frac{\text{Jupiter } (u_2)}{SD1(u)''} = \frac{33.75}{20.859375} = 1.6179775281 = \text{Fibonacci ratio } (144/89) \quad (12)_{J/DF2}$$

At which point Babylonian astronomy in general and the origins of these mean synodic arcs in particular begin to assume an unexpected level of importance despite almost universal dismissal of Babylonian methodology at the present time. For this reason the Babylonian observational reference frames and resulting luni-solar parameters in particular offer a minor introduction to the optional excursus at the end of Part 1.

V. Babylonian luni-solar parameters and *Phi*-series/synodic relations (1) and (2)

The inclusion of luni-solar parameters in the present context gives rise to the following *added* abbreviations, names, descriptions and periods in base-60 with decimal equivalents. All *bar* the tropical month and the tropical year were gleaned from leading authority O. Neugebauer's barely readable sexagesimal analyses rendered less understandable by the latter's non-model approach to Babylonian planetary theory. Because of these problems the following tables are largely *prior analytics* initially limited to mean values for synodic relation (1) subroutines applied in Table AP2

Abbr.	Astronomical Names and Standard Descriptions	Babylonian periods	Decimal days
MSM:	Mean Synodic month (new moon to new moon).	29;31,50,8,20 (rounded)	29.530594136 ^d
MSID:	Mean Sidereal month (fixed star to fixed star).	27;19,18 (rounded)	27.321666667 ^d
MTROP:	Tropical month (equinox to equinox; text, calc., added).	27;19,17,45 (rounded)	27.321574074 ^d
MAN:	Anomalistic month (perigee to perigee).	27;33,20 (unrounded)	27.555555555 ^d
MDRA:	Draconic month (node to node). ACT.	27;12,44 (rounded)	27.212222222 ^d
MDRA2:	Draconic month (node to node), ACT. calc.	27;12,43,59,40 (rounded)	27.212220679 ^d
SYR:	Sidereal year (fixed star to fixed star). calc. 12;22,8•MSM	365;15,38,17,44,26,40.	365.26063769 ^d
SYR2:	Sidereal year (fixed star to fixed star). calc. (MSM : MSID).	365;15,23,17,30.	365.25646991 ^d
TYRB:	Tropical year (equinox to equinox; (Bab. ACT 210, Sect.3)	365;14,44,51	365.24579167 ^d
AYR:	Anomalous year (perihelion to perihelion (calc., added).	365;15,34,18,22,58,51,	365.25952955 ^d
EYC:	Eclipse cycle (lunar node to lunar node) (text/mult/calc.).	(5458/465)•MSM.	346.61931784 ^d
AYC:	Anomalistic cycle (text/mult/calc; added)	(251/18)•MSM.	411.78772933 ^d

Table AP1. Astronomical terms, Babylonian mean luni-solar periods and decimal equivalents I

Next, the role played by the two primary *Phi*-series/synodic relations in the present context should also be noted:

$$\text{Synodic period } S_2 = \frac{T_1 \cdot T_3}{T_1 - T_3} \quad (T_1 > S_2 > T_3) \quad (1) \quad \text{Inner period } T_3 = \frac{T_1 \cdot T_2}{T_1 + T_2} \quad (T_1 > T_2 > T_3) \quad (2)$$

Significantly, the mean synodic month (MSM = T_1) and the tropical month (MTROP = T_1 and T_2) also play a role in the comparisons between other Babylonian luni-solar cycles, mean luni-solar periods and the modern values with variants of *Phi*-series/synodic relation (1) predominating in Table AP2.

#	Cycles and/or Periods	Subroutine ($T_1 > T_2 > T_3$)	Mean periods	Modern equivalents/; (sources)	Relations (1x)
[1]	Eclipse cycle (EYC)	(MSM : MDRA) =	346.619576 days	(Modern: 346.620107 days; (calc.)	(1e)
[2]	Anomalistic cycle (AYC)	(MSM : MAN) =	411.780405 days	(Modern: 411.783870 days; (calc.)	(1a)
[3]	Nodal cycle (days)	(MTROP : MDRA) =	6797.54400 days	(Modern: 6798.26051 days; Tables: 6798)	(1n)
--	Nodal cycle (years)	(MTROP : MDRA) =	18.6108756 years	(Modern: 18.6128373 years; (calc.,)	(1n)
[4]	Lunar perigee	(MAN : MTROP) =	3231.88186 days	(Modern: 3231.56072 days; Tables: 3232)	(1p)
[5]	Sidereal year (SYR, MYR)	(MSM • 12; 22,8) =	365.260637 days	(Expressed in mean synodic months = MYR)	--
[6]	Sidereal year (SYR2, calc.)	(MSM : MSID) =	365.256469 days	(Modern: 365.256365 days; Tables)	(1s)
[7]	Tropical year (TYR, calc.)	(MSM : MTROP) =	365.244059 days	(Modern: 365.242189 days; Tables)	(1t)
[8]	Tropical year (TYR, text)	(TYRB: 18-yr pd =	365.245792 days	365;14,44,51 days (ACT 210, Sec. 3)	--
[9]	Anomalistic year (AYR)	MSM • (360°/(u°) =	365.259529 days	(Modern: 365.259641 days; Tables)	$u = 29; 6, 19, 20^\circ$
[10]	SAROS, 19 EYC or	(223 MSM, calc.) =	6585.32249 days	(Modern: 6585.32163 days; Tables)	--

Table AP2. Astronomical terms, Babylonian luni-solar cycles and decimal equivalents II.

For example, although the slightly too large yet practical sidereal year SYR ([5]) is 12;22,8 mean synodic months or 365.260637 days, the more accurate value (SYR 2 = 365.256469 days) is readily available by way of synodic relation (1) utilizing the Babylonian mean synodic month (MSM = 29;31,50,8,20 days) and the mean sidereal month (MSID = 27;19,18 days):

$$\text{SYR2} = \frac{\text{MSM} \cdot \text{MSID}}{\text{MSM} + \text{MSID}} = 365.256469811878 \quad (365;15,23,17,28,45,43 \text{ days}) \quad (1\text{Syr2})$$

VI. The Tropical month from Babylonian luni-solar parameters

Also noteworthy are the extended Babylonian luni-solar cycles, especially those stated in eight lines of lunar text No. ACT 210, Section 3.²³ Although rarely recognized as such, they include one of the more contentious issues likely to arise in this context, *i.e.*, *presence of the Tropical month and the Tropical year* in Babylonian astronomy. The latter ([8] in Table AP2) occurs as "1,49,45,19,20 days of 18 years of the moon,"²⁵ yielding 354;14,44,51 days, which is superior to that used by Claudius Ptolemy (365;14,48 = 365.24666* days). More helpful, however, the presence of a tropical year supplies the means for determining a theoretical length for the *Tropical month* in Babylonian astronomy.

Applying a value for the Tropical year (TYRB) of 365;14,44,51 days and mean synodic month (MSM) of 29;31,50,8,20 days, an estimate for the tropical month (MTROP) is available from synodic relation (2) *i.e.*, subroutine TYRB : MSM:

$$\text{MTROP} = \frac{\text{TYRB} \cdot \text{MSM}}{\text{TYRB} + \text{MSM}} = 27.32160692 \quad (27;19,17,47,5, \dots \text{ days}) \quad (2\text{tr})$$

which rounds conveniently to 27;19,17,45 days and the even more convenient Babylonian estimate of 29;19,17,40 days for (perhaps) ACT 210 Section 3. The assignment of 365;14,44,51^d for a Babylonian tropical year was previously proposed by Hartner in an erudite discussion concerning *the tropical year and precession* which ended as follows²⁶

The inevitable conclusion to be drawn from the preceding demonstrations is, that in Babylonia under Achaemenian rule at the latest in 503 B.C., a clear distinction is made between the length of the tropical year: A = 365;14,48,33,37^d (possibly already then found exchangeable in practice with Ar = 365;14,44,51^d) and that of the sidereal year as underlying System B: PB' = 365;15;34, 18,1...^d (italics supplied)

Willi Hartner, "The Young Avestan and Babylonian Calendars and the Antecedents of Precession." *JHA*, X, 1979:1–22.

VII. Precession and the Babylonian Sidereal/Tropical years

Thus once again *Phi*-series/General synodic relation (2) is indicated, albeit with respect to mean values, whereas although the standard sidereal year [5] and tropical year [8] are both on the high side, their difference nevertheless yields a Seleucid Era value (perhaps known, perhaps not) for annual precession of 0;0,52,40,41,...^o and 24,602 years for the full cycle.

VIII The Anomalistic year

Unlike the derivations based on synodic relations *the anomalistic year* can be obtained from the *mean sidereal arc of Earth* (29;6,19,20^o) *per mean synodic month* of 29;31,50,8,20^d. This ratio yields a daily velocity of 0;59,8,9,43,22,7, ...^o

$$\text{Mean daily velocity of Earth} = \frac{29;6,19,20^o}{29;31,50,8,20^d} = 0;59,8,9,43,22,7, \text{ (0;59,8,9,43,20 rounded} = u') \quad (3m)$$

for a corresponding year of 365;15,34,18,22,58,51,40^d or 365.2595295... days. The modern estimate is 365.259641. . Or more simplistically, the amount moved by Earth along its orbit from one full-moon to the next. Thus, from ratio (3u) the mean period of Earth in mean synodic months is :

$$\text{Period of revolution of Earth} = \frac{360^o}{u'} = 12;22,7,51,53,40, \dots \text{ mean synodic months} = 365.25952955 \text{ days} \quad (3u)$$

IX. Multiple luni-solar extensions from *Phi-series/synodic relation* (1)

The simplicity of this relation permits similar derivations for the Draconic, Anomalistic and Nodal Cycles. The first pair include the mean synodic month (MSM) whereas the last cycle uses the Draconic (MDRA) and Tropical (MTROP) months:

$$\text{Draconic Cycle} = \frac{\text{MSM} \cdot \text{MDRA}}{\text{MSM} - \text{MDRA}} = 346.6195761217 \dots \text{ days} \quad (1d)$$

$$\text{Anomalistic Cycle} = \frac{\text{MSM} \cdot \text{MAN}}{\text{MSM} - \text{MAN}} = 411.7805352634 \dots \text{ days} \quad (1a)$$

$$\text{Nodal Cycle} = \frac{\text{MTROP} \cdot \text{MDRA}}{\text{MTROP} - \text{MDRA}} = 18.6101191842.. \text{ years} \quad (1n)$$

Here the nodal cycle is of potential interest in view of its association with lunar standstills in the first place and the apparent trouble the ancients took to delineate this phenomenon in the second, *e.g.* Chaco Canyon in the United States, Stonehenge in England and Callanish in Scotland.²⁷

At this point Babylonian astronomy in general and the origins of these mean synodic arcs in particular begin to assume an unexpected level of importance despite almost universal dismissal of Babylonian methodology at the present time. For this reason it appears necessary to offer an optional **excursus** after the Bibliography to explain the statements in Text-Fig 1 concerning advanced knowledge of astronomy in the Old Babylonian period and other matters of concern.

PART ONE: CLOSING REMARKS

Rejections: (1) Expansions of the Laws of planetary motion; (2) Benjamin Pierce's planetary framework

Starting with Galileo and the velocity expansions of the laws of planetary motion described in the Excursus, the concern here is that while the present writer was merely a tertiary restorer, and as such did not expect much in the way of applause, it seemed a reasonable assumption that the extended version $T^2 = R^3 = Vi^6$, $R = Vi^2$ would at least take its place next to Kepler's twin parameter format $R^3 = T^2$. And further, that variants of the former would simplify routine tasks, e.g., the calculation of angular momentum L , Table 1 mean velocities and the like. But this did not come to pass, and so it has remained ever since. On the other hand, modern science appears to have been able to function without such expansions, though not necessarily as well, it is suggested, had these velocity components also been incorporated.

But the real problem is not this historical item *per se*, but rather, that the same process and rapid dismissal was also applied to Benjamin Peirce's Fibonacci-based planetary framework with no replacement or improved version to take its place. And oddly, because of this situation which has remained unaddressed, humankind is now avidly searching for external planetary systems without any *overall dynamic understanding* of our own. Think not? Simply stated, no current model appears to exist which would, for example, provide the *precise* information and the *theoretical basis* for the possible existence of another planet interior to Mercury. Whereas, even in its initial form (*sans* intermediate intervals) this possibility was expressly incorporated in Pierce's initial approach, while in light of present concerns with Global Warming the possible *intermediate* location of Earth becomes more than a mere historical asterisk.

Weakened by special interests, discouraged by behavioural deficiencies and also impeded by disbelief, even the most fundamental question concerning whether Climate Change originates *primarily* from *within*, i.e., confines of planet Earth, or from *without* as an integral component of a larger System cannot be tackled adequately at present. Furthermore, what can be made of the location of Earth itself in the intermediate position between Venus and Mars, and what role might this apparent anomaly have played in global warming during the past, distant or otherwise?

All of which is further exacerbated by increasing population growth, unceasing deforestation, rapidly diminishing resources with warfare and mental illness also rising on a Global scale. Truly an Age of Disillusionment and concern.

In the meantime the present inquiry turns next to the initial application of the Pierce Divisor approach to external planetary structures with or without the following suggested guidelines.

PROVISIONAL GUIDELINES FOR EXTERNAL SYSTEMS

Test Format, *Phi*-series Relations and Base Periods

Remaining with the order adopted by Peirce which commences with the outermost PLANET #1 with the greatest period of revolution, moving inwards (by way of Synodic 2-1, then PLANET #2, etc.), will generally involve three consecutive mean periods. All of which can be determined by the following *Phi*-series synodic relations if needed:

Relation (1) The Synodic mean between two bracketing periods of revolution, thus the product of the periods divided by their difference.

Relation (2). Relation (2) requires *two adjacent periods above* to generate *the next value below*, and thereafter generates all further lower periods if or as required.

Relation (4). The geometric mean of any pair of bracketing periods. Thus Relation (4 ± 1) , or simply Relation (4) as used.

Relation(4E) Relation $(4E \pm 2)$, Relation $(4E \pm 3)$, Relation $(4E \pm 4)$, Relation $(4E \pm 5)$ and Relation $(4E \pm 6)$. Such applications depend on specific prior restorations (in due order) above and below the target position(s).

Relation(4F) Relation $(4F+3)$. Special case for PLANET #2 only. Requires both the Base period and periods below #2. This application serves to synchronize the restored periods at this point with the those of the divisor framework

The above relations are provided in Table 4 with the Fibonacci and the Lucas series in vertical and inverted form to match their inclusions in Tables 2a, 2b and also the format adopted for exoplanetary structures.

Lastly, possible departures from the framework are included as variations which may be encountered among external systems. For similar systems, however

- (a) Planets may occupy intermediate (synodic) locations, as in the case of Earth.
- (b) Planets and adjacent synodic locations may be unfilled (*i.e.*, absent), e.g., the Mars-Jupiter Gap.
- (c) Departures from the theoretical framework, or (a) and (b) may indicate disrupted planetary systems.
- (d) Planetary systems may possess residual Fibonacci indicators, as in the Solar System.
- (e) Planetary systems may also possess residual Lucas indicators for the same reason as (d).

Planets N Divisors Synodics # (added)		Phi-SERIES RELATION (1), the Synodic mean: $B = AC / (A - C)$. For any three successive Phi-series periods, A, B, C middle period (B) is the <i>product of the periods</i> on either side <i>divided by their difference</i> .	Fibonacci series 1, 1, 2, 3, 5	Lucas series 1, 3, 4, 7
PLANET 1 1 Synodic 2-1 1			1 4181	1 5778
PLANET 2 2 Synodic 3-2 4		Phi-SERIES RELATION (2), the Half-harmonic mean: $C = AB / (A + B)$. If two upper adjacent periods A, B are known, the third and lower (C) is the <i>product of the two adjacent periods divided by their sum</i> .	2 2584	3 3571
PLANET 3 6 Synodic 4-3 9			3 1597	4 2207
PLANET 4 15 Synodic 5-4 25		Phi-SERIES RELATION (4), the extended Geometric mean: $B = \sqrt{(AC)}$. For any three successive periods, the middle period (B) is the <i>geometric mean</i> of the periods on either side, as are the resulting periods for the positions $\pm 2, \pm 3, \dots = (4E)$; relation $(4F+3)$ pertains to Planet #2 alone.	5 610	7 1364
PLANET 5 40 Synodic 6-5 64			8 377	11 843
PLANET 6 104 Synodic 7-6 169		BASE PERIODS B1/B2. Base period B1 is the period of the outermost planet as detected. Base Period B2 may be applicable if Synodic 2-1 is marginally > B1.	13 233	18 521
PLANET 7 273 Synodic 8-7 441			21 144	29 322
PLANET 8 714 Synodic 9-8 1156		BASE PERIOD B3. Approximate base periods (B3s) result from <i>reversed procedures, i.e., the products</i> of known periods and their assigned divisors.	34 89	47 199
PLANET 9 1870 Synodic 10-9 3025			55 55	76 123
PLANET 10 4895		BASE PERIOD B4. Approximate base periods (B4s) can be obtained from the averaged values of the available B3 <i>products</i> .	89 34	123 76
			144 21	199 47
		BASE PERIOD B5. Working, non-standard base periods (B5s) can be applied when none of the above prove to be applicable.	233 13	322 29
			377 8	521 18
			610 5	843 11
			987 3	1364 7
			1597 2	2207 4
			2584 1	3571 3
			4181 1	5778 1
			Fibonacci series	Lucas series

Table 4. Divisor assignments, numerical series, Phi-series relations and conventions for base periods B1 thru B5.

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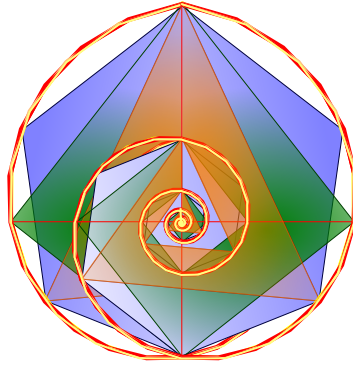
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INTIMATIONS OF COMMONALITY IN PLANETARY SYSTEMS

PART I.	The Pierce Planetary Framework (1850) Revisited. (1.272 Mb).
PART II.	The Pierce Framework and External Systems. (379 Kb).
PART III.	Real-time Motions in the Solar System and the Golden Ratio. (3.282 Mb).
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PART V.	Time and Tide: The Spiral Form in Time and Place. (15.75 Mb).
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OTHER:	PART I plus the EXCURSUS. (1.311 Mb).
MAIN:	Full text plus the EXCURSUS. (21.952 Mb)

PART ONE



EXCURSUS

HISTORICAL ISSUES & PROBLEMS
(OPTIONAL)

I. The Babylonian Ecliptic Reference Frame and the Zodiac

Apart from the relatively recent *Astronomical Diaries and Related Texts* by Abraham J. Sachs and Hermann Hunger (1988) this is still a neglected area. Yet there is literally nothing in the historical record which comes remotely close to details in mathematical cuneiform texts from the Old Babylonian Period (1900 BCE-1650 BCE), the Ephemerides, the various "Procedure" texts of the Seleucid Era (310 BCE - 75 CE) and the Babylonian Astronomical Diaries from 625 BCE - 65 BCE. Included among the latter are the two frames of reference against which the major astronomical phenomena take place; 1. The local horizon, and 2. A fixed celestial reference frame provided by + 33 bright stars distributed around the ecliptic, *i.e.*, within a few degrees of the paths of the visible planets.

The *Diaries* represent a prime resource while also including background information on the subject plus three sets of longitudes (*L*) and inclinations (*i*) of the Babylonian ecliptic reference stars for 600 BCE, 300 BCE and the Year 0 CE. For more than enough data, in fact, to display the zodiacal reference frame and the ecliptic "normal" stars with their various names and signs against those published in Neugebauer's *Astronomical Cuneiform Texts* (1955) to overcome the difficulties caused by the latter's overly condensed and far from helpful presentations of both the texts and their contents. Thus, starting with Figure 2a below, the dual names and signs for the Zodiac used in this major reference followed by a condensed presentation of Babylonian planetary parameters and periods relations on the next page.

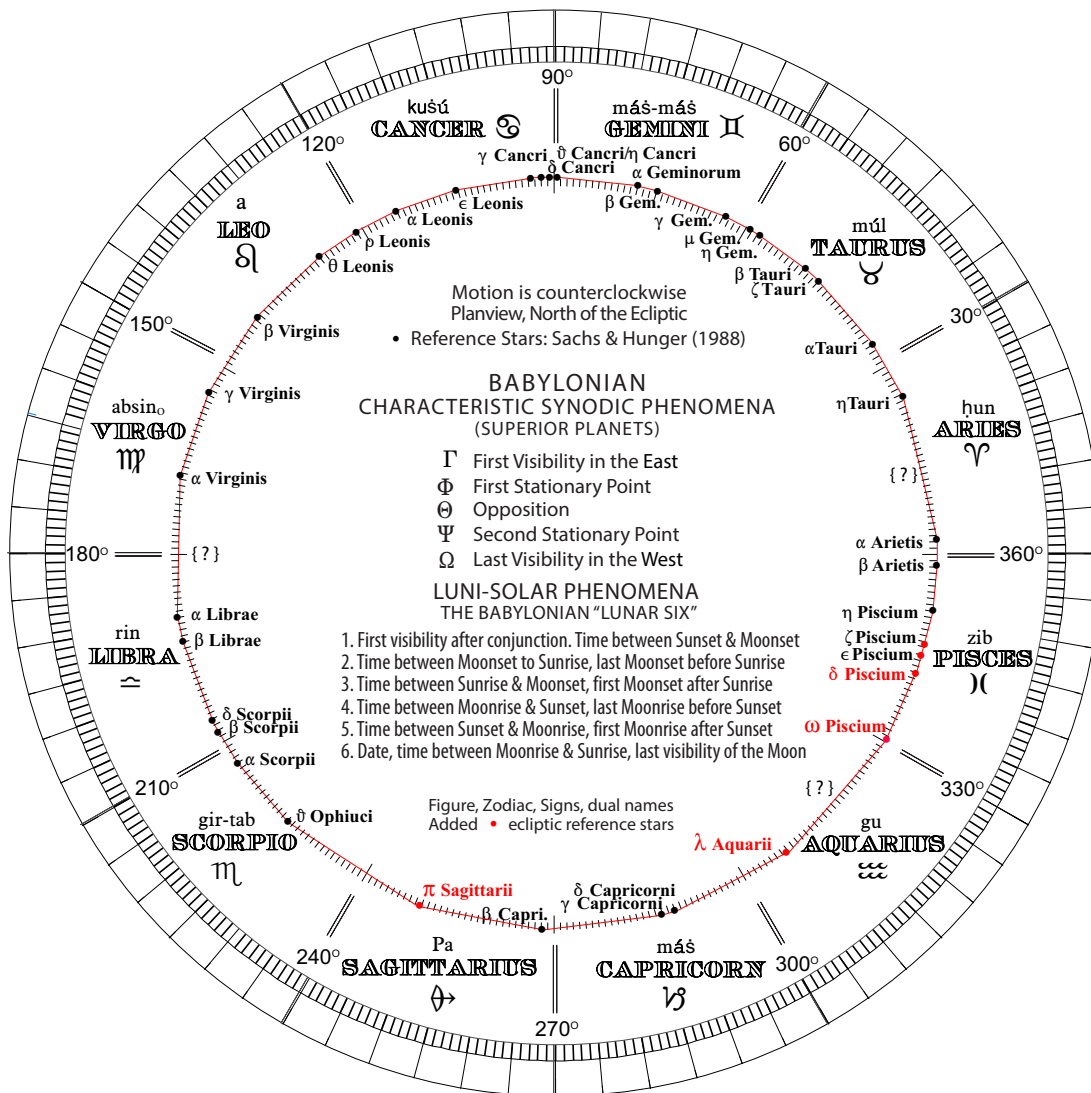


Fig. 2a. Babylonian ecliptic "Normal" Stars; the Zodiac plus standard signs and dual names

♈ Aries (hun)	0°	♌ Leo (a)	120°	♐ Sagittarius (Pa)	240°
♉ Taurus (múl)	30°	♍ Virgo (absin)	150°	♑ Capricorn (más)	270°
♊ Gemini (máš-máš)	60°	♎ Libra (rin)	180°	♒ Aquarius (gu)	300°
♋ Cancer (kušú)	90°	♏ Scorpio (gir-tab)	210°	♓ Pisces (zib)	330°

I. BABYLONIAN PLANETARY PARAMETERS AND PERIOD RELATIONS

The five planets known in Antiquity are shown in planview north of the ecliptic in heliocentric order with Earth (following Mercury and Venus) third planet from the Sun. Also supplied are modern estimates for the mean periods of revolution and calculated mean synodic periods/lap-cycles from relations (2s) and (2i) given on page 2.

A. Mean Periods of revolution (T)⁶

N ¹	PLANET Sidereal	Modern T revolutions ²	Babylonian revolutions	Babylonian Ratio: N/Z	Diff. (T) % Diff.
3	Saturn	29.423519	29.444444	(265/9)	-2.093%
4	Jupiter	11.856520	11.861111	(427/36)	-0.459%
5	[M-J Gap]	-	-	-	-
6	Mars	1.8807111	1.8807947	(284/151)	0.0084%
-	[Earth]	1	1	[1]	-
7	Venus	0.6151857	0.6151790 ²	(1151/1871)	0.0004%
8	Mercury	0.2408444	0.2408377 ²	(46/191)	0.0007%

¹ N Planets counted inwards from Neptune. ² Revolutions added.

B. Mean Synodic periods/Lap cycles (S)⁷

N	PLANET Synodic	Modern S Cycle (years)	Babylonian Cycle (years)	Babylonian Ratio: N/II	Diff. (S) % Diff.
3	Saturn	1.0351821	1.0351563	(265/256)	0.003%
4	Jupiter	1.0921105	1.0920716	(427/391)	0.004%
5	[M-J Gap]	-	-	-	-
6	Mars	2.1354462	2.1353384	(284/133)	0.011%
-	[Earth]	-	-	-	-
7	Venus	1.5986349	1.5986111	(1151/720)	0.002%
8	Mercury	0.3172531	0.3172414	(46/145)	0.001%

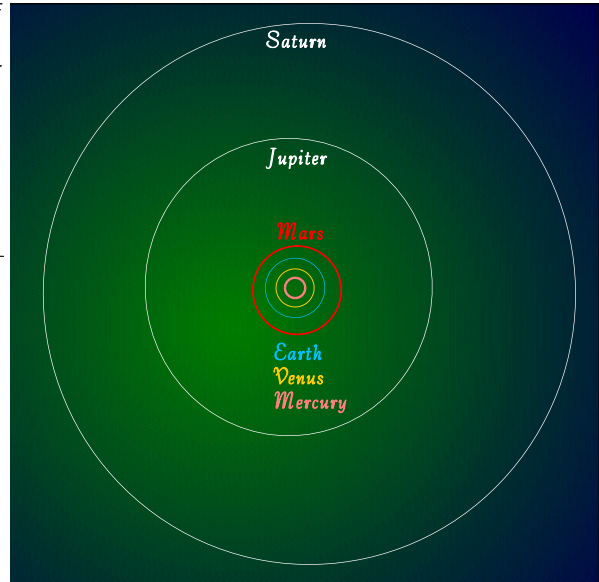


Table 4. Modern and Babylonian mean periods T , S in years **Fig.3.** The Orbits of the five planets known in Antiquity

Synodic relations (1), (2) and the modern equivalents all have roles to play in Table 4, but relation (1) in full has a further application arising from the inclusion of the mean synodic month in Babylonian planetary theory beyond mere calendaric considerations. More on this later.

Babylonian mean periods, on the other hand, are from data-sets which provide insights into their determination, initial lack of revolutions for the two inferior planets included. As they now stand and as presented in ACT (1955) by Neugebauer, Babylonian mean periods were, however, seemingly “simple” integer period relations in a dual form:²⁸

Superior planets: N years : II Synodic arcs/periods : Z Rotations;
Inferior planets: N years : II Synodic arcs/periods. (only)

N Years = II Synodic arcs + Z Rotations
II Synodic arcs in N Years (no rotations)

whereas, in contrast to Neugebauer’s treatment, from a fictive, heliocentric viewpoint and *revolutions* substituted for “rotations” plus the mean *periods of revolution* (Z) added for Venus and Mercury, a more complete and balanced arrangement is:

THE SUPERIOR PLANETS: SATURN, JUPITER AND MARS

SATURN: 265 years - 256 Synodic arcs = 9 Revolutions

JUPITER: 427 years - 391 Synodic arcs = 36 Revolutions

MARS: 284 years - 133 Synodic arcs = 151 Revolutions

THE INFERIOR PLANETS: VENUS AND MERCURY

VENUS: 1151 years + 720 Synodic arcs = 1871 *Revolutions (added)*

MERCURY: 46 years + 145 Synodic arcs = 191 *Revolutions (added)*

Also (shown later) four more period relations for Mercury are attested.

which yields by integer division results which are comparable to the modern values shown in Table 4.

II. Babylonian “characteristic” synodic phenomena

From a modern viewpoint it is easiest to start with Jupiter at Opposition to introduce the “characteristic” synodic phenomena since of the five Babylonian synodic events it is only at Opposition that the *location* of Jupiter is at its *true position*. Whereas, for three of the other four phenomena the locations are merely *apparent*, while for the fourth (from the Last Visibility in the West to the First Visibility in the East) Jupiter is moving on the opposite side of the Solar System and cannot be observed at all throughout the interval. For this reason the following figures include the annual *sidereal* motion for Jupiter with Figure 4a initially assigned an integer value of 12 years for the period of revolution (T) before including the Babylonian characteristic phenomena as an integral set in Figure 4b.

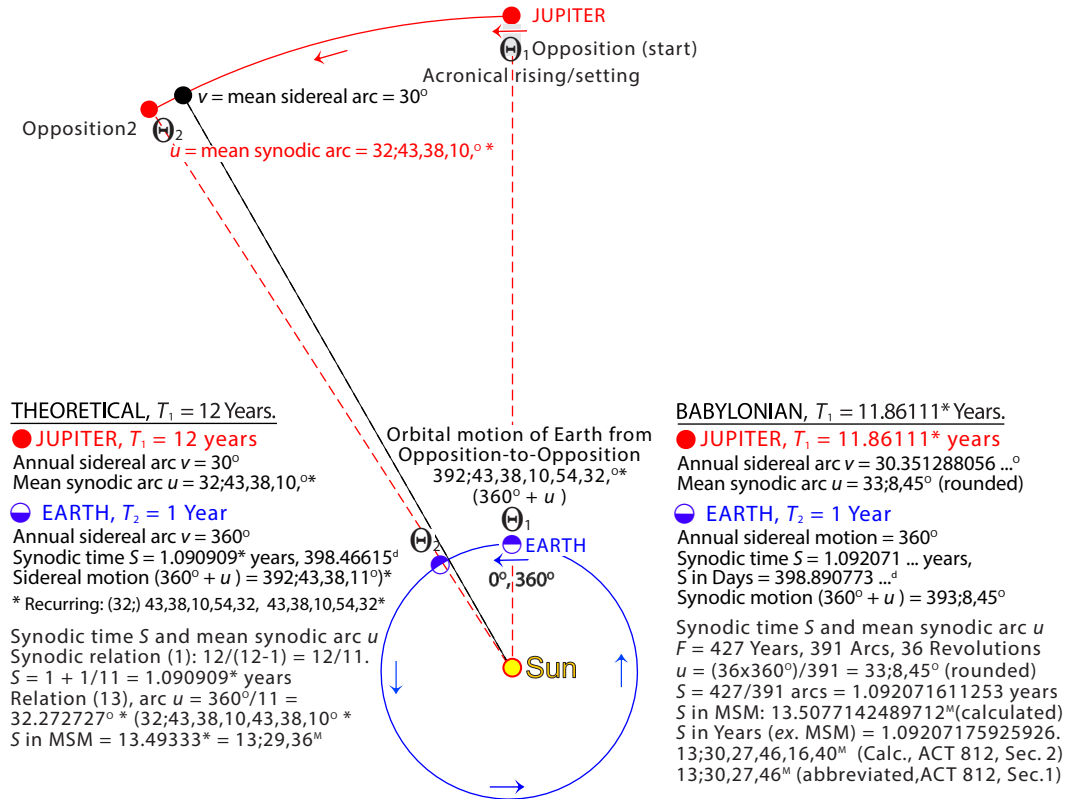


Fig. 4a. The mean synodic arc u for Jupiter, theoretical T_1 12 years. Babylonian T_1 as used 11.86111* years.

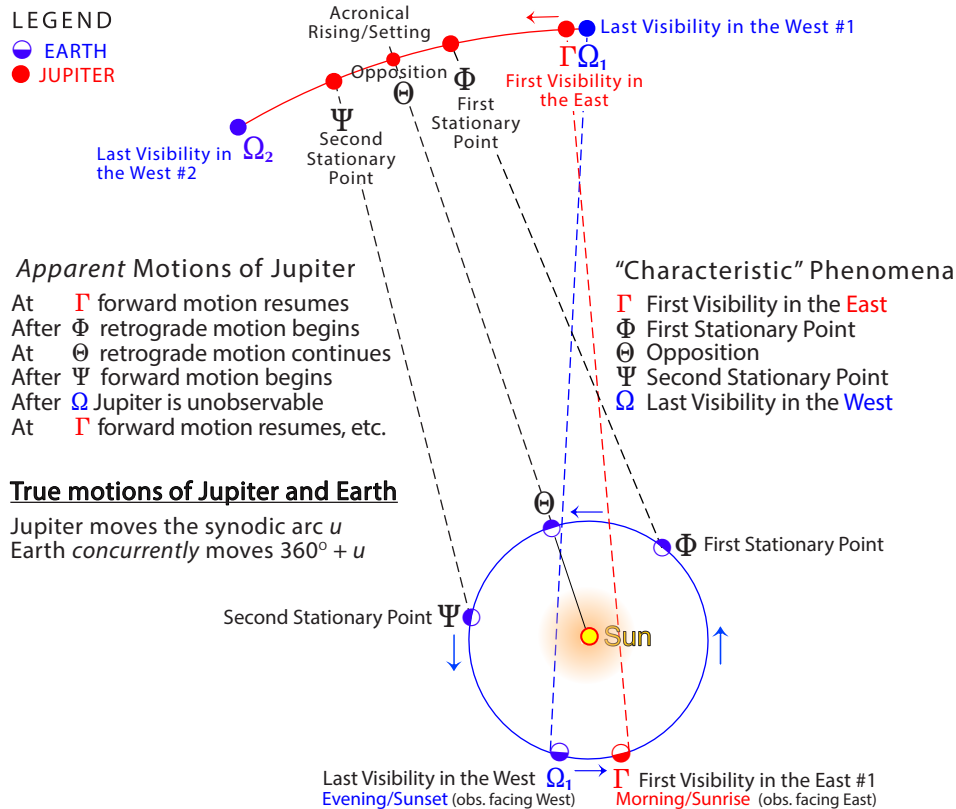


Fig. 4b. Jupiter: Characteristic Phenomena and concurrent heliocentric motion of Earth.

Shown in planview north of the ecliptic in Figure 6, the Babylonian reference frames enclose the “characteristic” synodic phenomena for Jupiter on the 34° synodic arc accompanied by the direct heliocentric motion of Earth. As perhaps stated in Inset A and Note 1:²⁹

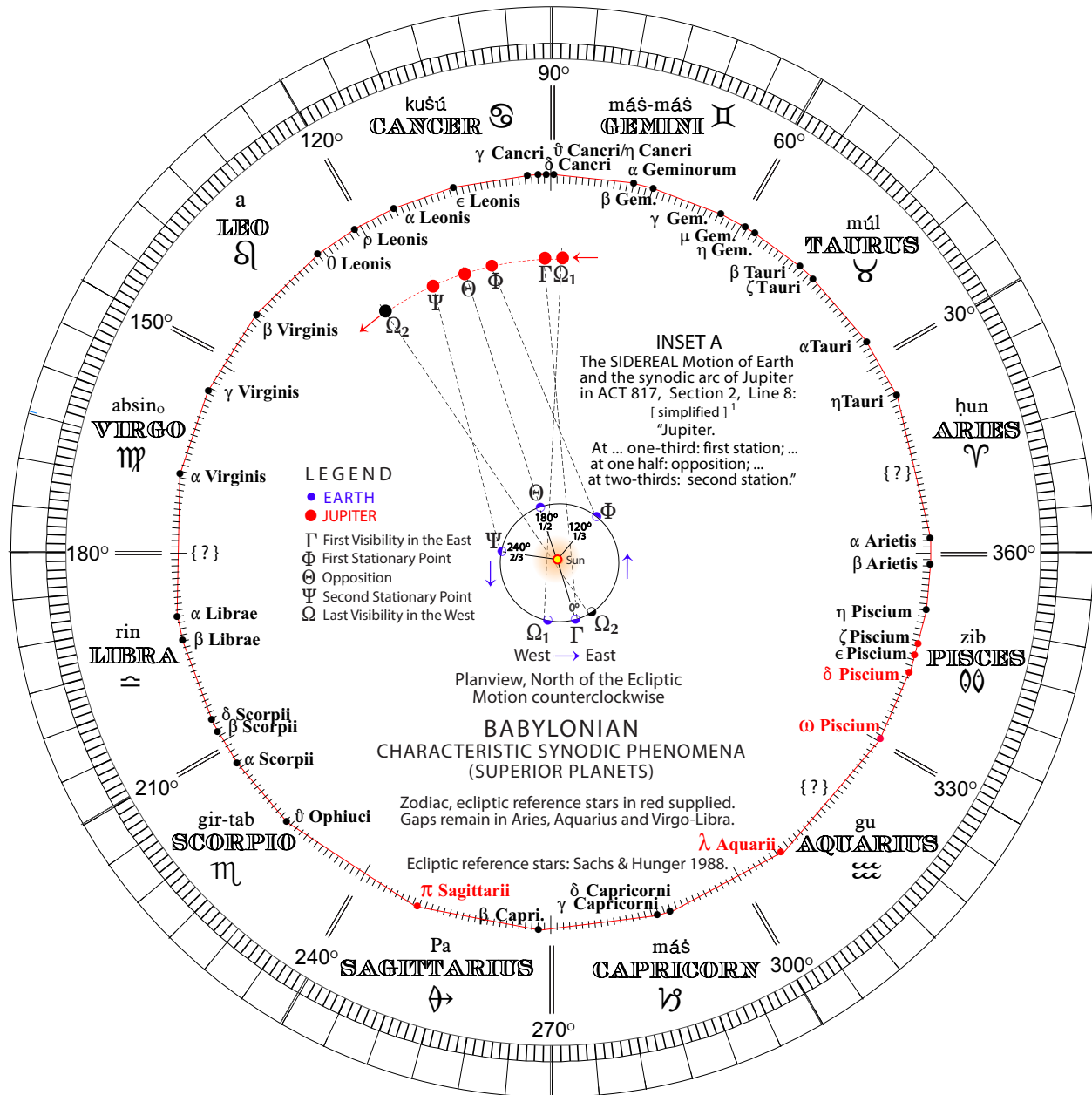


Fig 5. The Zodiac, Babylonian Ecliptic Reference Stars, Synodic Phenomena, Heliocentric motion of Earth.

¹ ACT 817, Section 2, Line 8 in full: "Dr. Sachs suggested the following interpretation of line 8: 'Jupiter. At one-third of its (stretch of) visibility: first station; at one-half of its (stretch of visibility): [Opposition; at two-thirds: second station]'. This is indeed in good agreement with the general schemes for subdividing the synodic arc of Jupiter." Otto Neugebauer, ACT (1955: 430).

But It is *EARTH* which is moving approximately “one-third,” then “one half,” followed by “two-thirds” over the full synodic arc of $360^\circ + u$. Plus the period for *Earth* to move from the Last visibility of Jupiter in the West (Ω_1) to the First visibility of Jupiter in the East (Γ) from ACT 813, Section 11 is 27 days (in Cancer ☉) and 32 days (in Sagittarius ♐).³⁰ Thus a mean value of 29.5 days for the *varying* intervals of invisibility *versus* the mean synodic month (29.5306 days) with regular variations arising from the elliptical orbits of Moon, Earth and Jupiter. Plus considerable problems from parallax to further complicate the precise location of Jupiter on the day of the First visibility in the East.

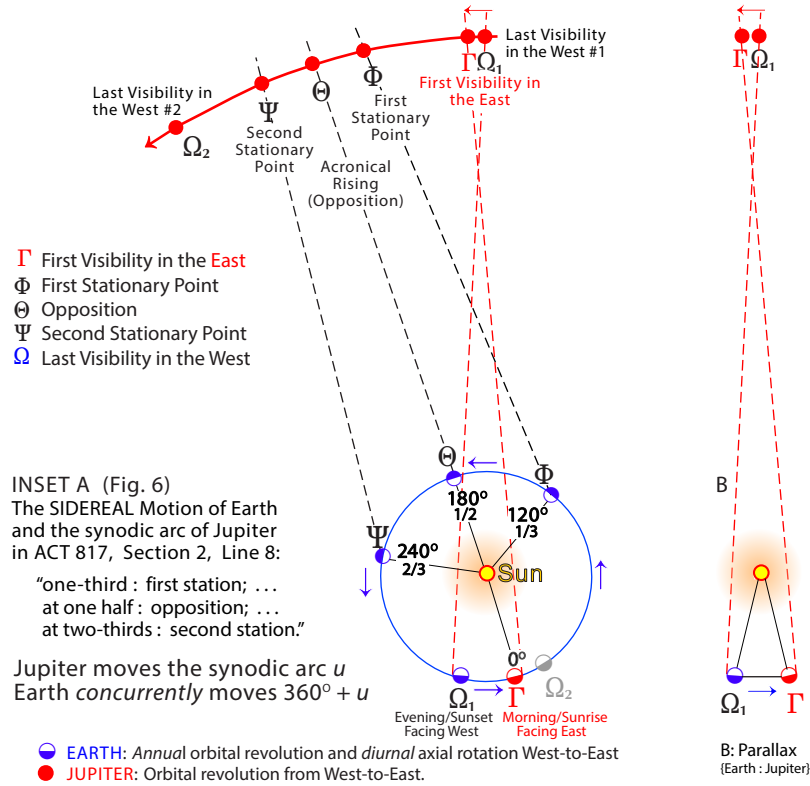


Fig. 6. Heliocentric motions of Jupiter and Earth from line 8, Section 2 of ACT 817.

III. The determination of the Babylonian Integer Period Relations

Shown in plan view north of the ecliptic it has long been recognised that the relatively small eccentricities of the Solar System planets cause the orbits to resemble slightly off-set circles as opposed to elliptical orbits *per se*.

What might all this have to do with Babylonian planetary theory? A great deal, as it turns out, because - as in the case of the mean periods - modern eccentricities also have an *a priori* element to them. In other words, the mean distances and the eccentricities are *given* with the apsidal distances calculated from this base with respect to an *already assigned line of apsides*. Thus the locations of the shortest (and fastest moving) radius vectors/distances and the longest (also slowest moving) radius vectors for each planet. Whereas the Babylonians, having determined the mean periods and associated mean velocities from a lengthy series of observations, next seem to have derived the values for not only for the extremal synodic arcs, but also precise locations for the same. In short, lines of apsides. But first, how were the integer period relations which provided the Babylonian periods (T , S) in Table 4a obtained?

Fortunately, the origin of the integer relations is generally well known, even if the impressive results are not. Or perhaps better stated, still largely driven into obscurity by partial and less than praiseworthy modern analyses that fail to mention the accuracy so obtained. Nor do repeated references to "rotations" help here either, or an inability to consider the significance of the difference in format for the final relations of the Superior and Inferior planets.

Babylonian "long" period relations

The Final period relation for Jupiter: In 427 years: 391 synodic arcs and 36 Revolutions

The method for obtaining the mean and varying velocities and the associated times is explained in great detail in the procedure texts from the Seleucid Era, with those for Jupiter among the most informative, *e.g.*, as in ACT 813, a Babylonian "procedure" text for the determination of the *mean* synodic arc which in Sections 1 and 21 includes the simple integer period relation for Jupiter of 427 years as the end product:³¹

[in 6,]42 days from (one) appearance to the (next) appearance (Jupiter) moves 30 degrees.' This is correct for the slow arc of System A' for Jupiter. For lines 4 and 5 Sachs suggests: "Compute(?) for the whole zodiac (or for each sign) the appearance according to the day and the velocity.' This is followed in lines 5 and 6 by the rules for the periods: 'in 12 years you add 4;10, in 1,11 (71 years) you subtract 5, in 7,7 (427 years) the longitude (returns) to its (original) longitude'... ' In the Almagest the correction for 71 years is - 4;50° (IX3, Heiberg p. 215; Manitius II p. 100.

The translation and reference to the *Almagest* are by Neugebauer; decimal periods have been added for clarity and “revolutions” have also been substituted for “rotations” uniformly promoted by the latter. In passing it is safe to say that the latter replacement(s) largely correct Neugebauer’s generally inadequate terminology.

Origins and Implications

It is apparent from this procedure that for Jupiter a pair of integer periods were selected (named here *T1* and *T2*) with corresponding near-integer numbers of synodic periods, near-integer numbers of revolutions with *small and convenient corrections in longitude of opposite sign*. From these periods and corrections the final Babylonian “long” period integer relation of 427 years is readily obtained by addition. As in fact suggested by the writer in a *Letter to the Editor of ISIS* published in 1977³² in an earlier attempt to close Babylonian orbits about the common center.

T	Periods (Years)	Progress T1 - T2	Synodic Synodic arcs	Sidereal Revolutions	Corrections (Longitude)	
T1	12 years	(12)	11	1 x 360°	Slow +4;10°	Fast +5;00°
T2	71 years	(6*12)	65	6 x 360°	-5;00°	+6;00°
T3	83 years	(12 + 71)	76	7 x 360°	- 0;50°	-1;00°
T4	95 years	(12 + 83)	87	8 x 360°	+3;20°	+4;00°
T5	166 years	(71 + 95)	152	14 x 360°	-1;40°	-2;00°
T6	261 years	(166 + 95)	239	22 x 360°	+1;40°	+2;00°
F7	427 years	(166 + 261)	391	36 x 360°	0;00°	0;00°

Table 5. Babylonian Jupiter corrections and the final 427-year period **F**.

Although Neugebauer was aware of the corrections and almost all of the periods which lead to **F7**, the final integer period relation, he did not pursue the matter further, although (on inspection) closed orbits are obviously involved since each correction is also *heliocentric* in nature. For example, by the same process used for the *synodic arc*, the *annual sidereal arc* for Jupiter is obtained from the division of the total sidereal motion (as before $36 \times 360^\circ = 12,960^\circ$) by 427, the number of years for **F**, the final period. The *mean sidereal motion* for Jupiter (30.351288056° per annum) applied to the *T1* and *T2* periods gives the excess over 12 years (*T1*) and the deficiency after 71 years (*T2*) *modulo* 360° which results in values of 4.2152457° and -5.058548° ($4;12,55,38$, and $-5;3,30,46$), thus rounded to the nearest sixth of a degree ($0;10^\circ$) for both, values of $4;10^\circ$ and $-5;00^\circ$ as used by the Babylonians.

The 427-year Final period for Jupiter

Clearly, from the above the initial stage of the Babylonian inquiry appears to have been the derivation of an integer number of years to which there corresponded an integer number of mean revolutions and a corresponding integer number of mean synodic periods. With this primary goal for Jupiter achieved, the next step following in Section 21 of ACT 813 is the determination of the mean synodic arc (*u*) as follows:³³

[7,7 years (correspond to) 6,31 appearances (Γ),] 36 rotations, 3,36,0° motion, 33;8,[4]5
(is the) mean value of the longitudes,

which in plain English and decimal notation is clearly the unambiguous statement that:

427 years correspond to 391 synodic periods, 36 *revolutions*, $12,960^\circ$ *sidereal* motion, 33;8,45
(is the) mean value of the synodic arc. (added: with $33;8,44,48,29,^\circ$. . . rounded to $33;8,45^\circ$).

Thus, unequivocally, 36 revolutions of mean circular motion with a subdivided arc subtended from the center with 427 years also the 427 revolutions of Earth. And as seen next, the same argument is applicable to the division of the total sidereal motion of Saturn by the corresponding number of mean synodic periods to derive the mean synodic arc for this planet in the same manner. Except here the result is exact and convenient in both base-60 *and* base-10.

The 265-year Final period relation for Saturn

The identical procedure is given for Saturn in abbreviated form with the total sidereal motion supplied in Section 4 of ACT 802, a Saturn procedure text. Which in decimals with *revolutions* substituted for “rotations,” is:³⁴

Concerning Saturn. 265 years corresponds to 256 synodic periods, 9 sidereal revolutions and $3,240^\circ$ total motion.

leading to the mean synodic arc for Saturn *u* ($12;39,22,30^\circ$) from the total sidereal motion ($9 \times 360^\circ = 3,240^\circ$) divided by the number of mean synodic periods (256). Section 5 supplies the constants for the extremal synodic periods, and Section 6 the corresponding extremal synodic arcs. The two intermediate periods *T1* and *T2* with corrections of opposite sign for Saturn are 29 years and 59 years which lead in turn to the 265-year integer relation **F** of Saturn.

The 284-year Final period relation for Mars

For Mars $T_1 = 47$ years, $T_2 = 79$ years with the Final integer period $F_n = 284$ years, to which correspond $151 \times (360^\circ)$ periods of revolution and 133 mean synodic arcs/mean synodic periods. The total motion is again divided by the number of synodic events ($151 \cdot 360^\circ = 54,360^\circ$) divided by 133 = 408;43,18,29,46, . . . abbreviated to 408;43,18,30^o and ultimately $u = 48;43,18,30^\circ$. This differs markedly from Jupiter and Saturn where Earth moves $(360^\circ + u)$ in the time that it takes for these two outer planets to move the mean synodic arc u alone. Now, to lap Mars, Earth must move $(2 \cdot 360^\circ + u)$ during which time Mars itself moves $(1 \cdot 360^\circ + u)$. This departure from the format for both Saturn and Jupiter is perhaps one reason for earlier uncertainty concerning any Babylonian System B velocity function for Mars despite hints in Mars procedure text ACT 811a.³⁵

IV. The Inferior planets Venus and Mercury.

Venus

In addition to the well-known but barely recognized *Fibonacci* pairing of 5 synodic periods/arcs in 8 years, a longer Final period of 720 synodic periods/arcs in 1151 years is also attested. The number of revolutions for this planet are again absent, with the final integer relation merely the numbers of years and corresponding numbers of synodic arcs/periods:³⁶

Venus: 8 years corresponds to 5 synodic arcs/periods
Venus: 1151 years corresponds to 720 synodic arcs/periods

with the ratio of the two parameters providing the mean synodic period(s) in years: $8/5 = 1.6$, and more accurate $1151/720 = 1.5986111$ years. Expressed in days the latter mean synodic period for Venus is 583.9097^d versus the modern estimate of 583.9214^d.

Mercury

Five Babylonian period relations for Mercury are attested with 145 synodic periods in 46 year the best known.³⁷ As in the case for Venus the corresponding number of revolutions is not stated, an *apparent* deficiency maintained by Claudius Ptolemy in the *Almagest* and a mistake also retained by almost all subsequent commentators.

The resulting mean synodic period for Mercury is quite accurate, i.e., $46/145 = 0.317241$ years and 115.8758 days versus the modern estimate of 115.8775 days. The same degree of accuracy is also provided by the four remaining integer period relations. The latter set of limited Babylonian period relations for Mercury replete with Greek-letter assignments provided by Neugebauer are included below with the missing numbers of sidereal revolutions and the missing revolutions ($Z = T$) supplied.³⁸ The period of revolution (T) for the 46-year, 145 synodic period relation (and corresponding 191 revolutions) is 0.24083769 years ; the modern estimate for T is 0.24084445 years.

- Σ In 388 years, 1223 disappearances as a morning star (and 1611 revolutions, $T = 0.24084419$ years)
- Ξ In 480 years, 1513 appearances as an evening star (and 1993 revolutions, $T = 0.24084295$ years)
- Ω In 217 years, 684 disappearances as an evening star (and 901 revolutions, $T = 0.24084350$ years)
- Γ In 848 years, 2673 appearances as a morning star (and 3521 revolutions, $T = 0.24084067$ years)

Thus the 46-year period relation for Mercury does not even provide the best result, nor in their original Babylonian context do any of the *integer* period relations require a "correction," despite the later "improvements" by Claudius Ptolemy carried forward with the latter's defective geocentric planetary model to the time of Copernicus.

VI. The significance of the unstated periods of revolution for the Inferior planets

Although the final period (F) differs in format for the two Inferior planets from those applied for Mars, Jupiter and Saturn, the lack of stipulated numbers of periods of revolution for Venus and Mercury is nonetheless merely a function of the heliocentric nature of Babylonian planetary theory. In other words, though the motion of Earth provides the frames of reference for both the times and the velocities, it is always the outer planet of the respective synodic pair that provides the synodic arc. For the superior planets it is Earth which laps Mars, Jupiter and Saturn during each synodic period, i.e., each lap-cycle:

SUPERIOR PLANETS

From the Number of Years, corresponding numbers of Synodic arcs/periods and numbers of Revolutions
Outer and Slower-moving SATURN, and JUPITER move the direct (synodic) arc (u).

Inner and Faster-moving EARTH concurrently moves $(360^\circ + u)$ to supply the time per lap-cycle.

Mars is the exception where Earth moves $(2 \cdot 360^\circ + u)$ during which time Mars moves $(1 \cdot 360^\circ + u)$.

1. EARTH in all cases supplies the mean synodic time per lap-cycle, but not the mean synodic arc (u).

INFERIOR PLANETS

From the relation: Number of Years and corresponding number of Synodic arcs/periods only

Outer and Slower-moving EARTH moves the direct (synodic) arc (u).

while inner and faster-moving VENUS and MERCURY *concurrently* move ($n \cdot 360^\circ + u$) per lap-cycle.

1. EARTH supplies the mean synodic time per lap-cycle. 2. EARTH also supplies the mean synodic arc (u).

VENUS: In 1151 years, 720 synodic cycles (and unstated: 1871 revolutions)

Mean Synodic time: $1151/720 = 1.59861111^*$ (1;35,55) years, or 583.909714 days.

Mean synodic arc (u): $(1151 \cdot 360^\circ / 720) = 575.5^\circ$ and/or 215.5° from $(1 \cdot 360^\circ + u)$.

during which time VENUS *concurrently* moves $(2 \cdot 360^\circ + u)$.

MERCURY: In 46 years, 145 synodic cycles (and unstated: 191 revolutions)

Mean Synodic time: $46/145 = 0.3172413793$ years or 115.8757885 days.

Mean synodic arc (u): $(46 \cdot 360^\circ / 145)$, $u = 114.2068965^\circ$.

during which time Mercury *concurrently* moves $(1 \cdot 360^\circ + u)$.

Thus from the heliocentric viewpoint, for the inner planets Earth is now supplying the synodic arc as an outer planet, but still providing the synodic time as before. Thus a variant of the known relation for the Superior planets where the number of revolutions are components of the integer relation itself. Therefore, remaining with the terminology and the symbols used by Neugebauer (albeit with *revolutions* substituted for *rotations*), the primary Superior planet relation necessarily includes the following re-arrangement of terms:

SUPERIOR PLANETS. Variant of the primary period relation for Saturn, Jupiter and Mars:

Number of Sidereal Revolutions (Z) = Number of Years (N) - Number of Synodic periods (II)

e.g., returning to Jupiter, 36 revolutions = (427 years-391 synodic periods) with the mean synodic arc (u) = $33;8,45^\circ$ derived and rounded as given in ACT 813, Section 21. Although not needed for practical reasons, should the actual periods of revolution for Venus or Mercury be required they can be readily obtained from the following variant:

INFERIOR PLANETS. Variant of the primary period relation

Number of Sidereal Revolutions (Z) = Number of Synodic periods (II) + Number of Years (N)

Thus for Venus the Fibonacci series itself, i.e.,

5 Synodic periods + 8 Years = 13 sidereal revolutions ($5 + 8 + 13$, etc.),

with the mean period of revolution (Z) for Venus from: $8/13 = 0.615385$ years, versus:

1. The modern estimate = 0.615186 years
2. Venus from $1151/1871 = 0.615179$ years
3. The reciprocal of Φ = 0.618034 years

VII. Babylonian astronomy: a modern enigma

Lastly, although the astronomical assignments for the Old Babylonian Period in Text-Fig 1 are at odds with prevailing chronology and understanding they are not made lightly or without supporting research and textual support.

Nevertheless, as far as Babylonian astronomy is concerned, primary modern works by Otto Neugebauer (1955,³⁹ 1975⁴⁰), Bartel van der Waerden (1974)⁴¹ and Noel Swerdlow (1998)⁴² accept the odd premise - despite procedures and details in the astronomical cuneiform texts - that the Babylonians possessed no fictive planetary model at all.

In fact, by the time of his final *opus*, *A History of Ancient Mathematical Astronomy* (1975) Neugebauer states with ill-founded certitude that the understanding of Babylonian astronomy :⁴³

is a rather self-contained problem of great technical complexity that will occupy us in the following chapters. *But it will reveal to us the working of a theoretical astronomy which operates without any model of a spherical universe, without circular motions and all the other concepts which seemed "a priori" necessary for the investigation of celestial phenomena. The influence of the available mathematical tools on the development of a scientific theory is perhaps far greater than we customarily admit.*

Otto Neugebauer, *A History of Mathematical Astronomy* (1975:347-348; italics supplied)

Unfortunately, this nihilistic, pre-conclusive viewpoint in consort with base-60 mathematics, unfamiliar procedures, hard to follow names and concepts has rendered the Babylonian material difficult to read in the first place and not worth reading in the second. Hence the unwarranted yet still prevailing ignorance on the matter.

Quite literally, Neugebauer "wrote the book on the subject," i.e., *Astronomical Cuneiform Texts* published in 1955, but this last assessment is not only erroneous, it also leaves one wondering how Neugebauer managed to escape critical peer reviews, the opposing discussion of precession and Babylonian years by W. Hartner (1979)²⁶ excepted. Yet according to Neugebauer Babylonian astronomy exists without "any model" and "without circular motions and all other concepts . . . necessary for the investigation of celestial phenomena," which is readily countered by the

Babylonian methods for the determination of fundamental period relations and mean synodic arcs provided in the various "procedure" texts of the Seleucid Era. Except that this material - replete with commentaries and notes - is in Neugebauer's *Astronomical Cuneiform Texts* (1955), and furthermore, key Babylonian procedures in this initial work either remain erroneous or incomplete. Whereas a further impediment to understanding - also introduced by the latter in his latest compendium - was a far from helpful caution, namely that: ⁴⁴

In working with sexagesimally written texts it is essential not to convert the numbers to decimals, to carry out operations decimally and only to change results back to sexagesimals . . . roundings in one system do not mean the same in the other and accurate parameters given in sexagesimals may be altered by the transition through decimal computations.

Otto Neugebauer, *A History of Mathematical Astronomy* (1975:1113)

Thus Neugebauer's unfathomable rejection of circular motion and further insistence on sexagesimal mathematics for Babylonian astronomy. But his earlier closing remark concerning mathematical tools was not only incorrect, it was also decidedly unwise. In fact, it is the height of irony that around the time HAMA was published (1975), Robert R. Newton, using a 10-decimal place pocket calculator put an end to the entirely unmerited reign of the *Almagest* by showing that the latter's supposedly accurate daily velocities for the planets (each given by Ptolemy to the *sixth* sexagesimal place) were *all erroneous*.⁴⁵ And therefore, they no longer "saved the phenomena," as claimed, while Otto Neugebauer, having spent his illustrious career heaping praise on Claudius Ptolemy and the *Almagest*, was obliged to admit in a feeble footnote in HAMA that he too had never checked the accuracy of the daily velocities in the chronically flawed *Almagest*.⁴⁶ More to the immediate point, it also turns out that the velocities so used were in all likelihood appropriated from limited, *earlier* sets of completely misunderstood Babylonian period relations.

It could be said that awareness of these errors only came to light with the publication of Robert R. Newton's *The Crime of Claudius Ptolemy* (1977)⁴⁵ and his *The Origins of Ptolemy's Astronomical Parameters* (1982),⁴⁷ but this is not entirely so. Both Al-Bitruji (ca. 1200)⁴⁸ and Copernicus (ca. 1500),⁴⁹ using for the most part the same period relations as Ptolemy were also unable to obtain compatible results, although the former, with a liberal sprinkling of the word "about," at least forewarns the reader of the uncertainties involved.

The main difficulty, it would seem, is not so much *geocentricity*, but restrictions imposed by having to incorporate *uniform circular motion* in association with basically *unworkable* auxiliary devices. Plus one further restriction - *also never met with Ptolemy's values* - which was that the sums of the supposed velocities in "anomaly" and "longitude" in each and every case were required to *equal* the velocity of the Sun.

One can certainly work through all the period relations employed by Ptolemy, Al-Bitruji and Copernicus to check the disparities discussed by Robert R. Newton, although it is nevertheless an irritating task. At present it is sufficient to note that as recently as the second half of the Twentieth Century - Manitius⁵⁰ (1963 reprint), Pedersen (1974)⁵¹ and even Neugebauer (1975)⁵² were unaware of the multiple errors in the daily velocities in Ptolemy's *Almagest*.

Perhaps Neugebauer was encouraged and emboldened by lack of criticism for earlier omissions in ACT (1955), not to mention its unreadable and disjointed presentation, but for whatever the reasons, even passing knowledge of the Babylonian procedures in the astronomical cuneiform texts of the Seleucid Era - the primary source for both ACT (1955) and HAMA (1975) - indicates that the Babylonian methodology routinely and indisputably employed uniform circular motion for the *initial* derivation of the *mean* synodic arcs for Sun, Moon and the planets. But unlike the Ptolemaic planetary model which utilized uniform circular motion and auxiliary devices to account for observed variations, the Babylonian used *mean circular motion* for the *mean synodic arcs* as the basis for schemes to account for variations *on either side* of established mean values. Circular motion is certainly applied in Babylonian astronomy but as an *intermediate step* towards the determination of apsidal velocities (predominantly System A) with more detailed schemes to determine the *varying velocities* and the associated *varying times* in System B. All of which was achieved before the time of Hipparchus (ca. 150 BCE) and well before Claudius Ptolemy's much-praised *Almagest*. Furthermore, the same fractional parts of misunderstood Babylonian period relations were, it would seem, not only applied by the latter, but also used in the alternate *geocentric* planetary model by Al-Bitruji (ca. 1200) and ultimately in the *heliocentric* model proposed by Nicholas Copernicus (ca. 1500).

VIII. Babylonian astronomy during the 20th Century

Then how did the current mindset which holds that the Babylonians themselves possessed no planetary model or any fictive understanding become established? There are numerous issues in ACT for which corrections are long overdue with unknown parameters and operations in the Babylonian procedure texts in need of explanation. Some deficiencies are elementary enough, but they should surely have been fully addressed by now, specially for Jupiter. This is clear from the distribution of the planetary texts in ACT cataloged by Neugebauer in his Introduction to the Planetary Theory in General: ⁵³

Our material contains ephemerides for all five planets. The distribution is very uneven: 41 texts and fragments for Jupiter, but only 12 for Saturn, 11 for Mercury, 9 for Venus, and 8 for Mars. In quoting these texts, we adopt the following notation:

Nos. 300 ff. Mercury
Nos. 400 ff. Venus
Nos. 500 ff. Mars
Nos. 600 ff. Jupiter
Nos. 700 ff. Saturn
Nos. 800 ff. Procedure texts

The “procedure texts” also deal with all five planets but here, too, Jupiter is better represented than all the other planets together.

The foundation of our understanding of the planetary texts was laid by Kugler¹ and Pannekeok.² Later contributions were due to Schnabel,³ and van der Waerden.⁴ Our material seems to be large enough to justify the statement that the planetary theory was less developed than the theory of the moon. We find, e.g., no direct consideration of the latitude. The complications of the planetary motion on one hand, the lack of of calendaric importance on the other might be the reasons for this fact.

It must not be forgotten, however, that our knowledge of Babylonian planetary theory rests on a much smaller basis than the lunar theory. Not only is the total number of texts for all five planets less than half of the lunar texts, but also the state of preservation of the procedure texts is much worse in the case of the planets than for the moon.

Otto Neugebauer, *Astronomical Cuneiform Texts* (1975:279)

The inclusion of “procedure” texts is an initial surprise, as is the larger luni-solar component with its potential for providing insights concerning the Babylonian approach to Earth/Sun dualities. This is the essential matter that lies at the heart and soul of all Solar System planetary theories.

So far Neugebauer’s introduction to the Babylonian material is encouraging as well as intriguing, and the same can be said concerning the cautions included in his explanation of “The Leading Ideas of the Planetary Theory.”⁵⁴

The discussion of the Babylonian planetary theory which follows is not intended to give a historically or astronomically complete description of this field. Its only purpose is to prepare the reader for the study of the texts themselves, that is, to enable him to compute the ephemerides with the methods used by the ancients and to grasp the general trend in the procedure texts. For all details, however, he must consult the texts themselves plus the notes and commentaries added to the translations and the relevant remarks in the Introductions concerning the theory of the single planets. The reader must remember what I have said in the preface: this is only an *edition* of the texts, not a systematic study of the contents.

Otto Neugebauer, *Astronomical Cuneiform Texts* (1955:279)

But if ACT (1955) was indeed “only an edition of the texts, not a systematic study of the contents,” then without further analysis, how did the notion that this material was insufficient to support a fictive planetary model arise? A partial answer would seem to be that *prior analytics* with initial emphases on available tools - units of time and measure, astronomical reference frames, mathematical methodology and results as recorded in the astronomical cuneiform texts - were not carried out adequately. Yet the latter seem to be relatively limited sets of data, i.e., the cumulative progress of the planets in terms of varying synodic arcs with accompanying synodic times, in short, to⁵⁵

find the longitudes and dates for the consecutive “characteristic phenomena” of the planetary appearances, such as helical risings and settings, oppositions, stations, etc. Only if these phenomena are known, can the location of the planet be determined for intermediate moments by means of complicated interpolations. But the overwhelming majority of the “ephemerides” and the procedure texts are devoted to the problems of determining the sequence of the characteristic phenomena, whereas the problem of “daily motion” is of a secondary character.

Otto Neugebauer, *Astronomical Cuneiform Texts* (1955:279)

The “secondary” nature of daily motion is not a problem if it is given due consideration. It does, moreover, supply a reminder that *the diurnal axial rotation of Earth* necessarily plays a fundamental role in the various “risings,” “settings,” “disappearances/last visibilities in the West” and “appearances/first visibilities in the East” of the synodic phenomena. Not alone, however, but with the daily *rotation* of Earth about its axis while simultaneously completing its annual orbital *revolution* about a common center, and in the same way the longitudinal progress of each planet. But it is here that Neugebauer’s description of Babylonian synodic phenomena takes a decidedly strange turn. First of all, although he is willing to concede that the Babylonian approach “is obviously the ‘natural’ one,” he nevertheless follows this reasonable statement with an historical aside that negates it while elevating later Greek achievements. Which in the case of Ptolemy and the vaunted *Almagest* turns out to be quite unmerited. Thus in full Neugebauer asserts that:⁵⁵

The Babylonian approach is obviously the ‘natural’ one.’ What one realizes first about the planets is their appearances and disappearances in the nightly sky, their stations and retrogradations. To predict these phenomena seems to be a real problem and it was solved by our texts by means of very ingenious arithmetical devices. But it marks an enormous step forward to ignore the ‘natural’ problems altogether and to ask an apparently much more complex question: how to describe the planetary motion as a whole. It is this shift of emphasis which led Apollonius, Hipparchus, and Ptolemy to their enormous successes.

Otto Neugebauer, *Astronomical Cuneiform Texts* (1955:280)

Unfortunately, most new readers would accept this assessment without question, coming as it does from noted Historian of Science Neugebauer who is also the foremost authority on Babylonian astronomy. Even so, from what he has said earlier about the limited and specialized nature of the Babylonian approach it would be premature to embark on such comparisons. But more to the point, despite Neugebauer's proclamations and denigrations, the instructions to his readers to consult the contents of the Babylonian texts provides the means to test whether the Babylonian methodology does (or does not) describe "planetary motion as a whole." And also - a necessary aside- why would anyone conclude that it did not, with the form of their approach of secondary importance, even if it is almost self-evidently *heliocentric* ?

The main problem, it seems, is that Neugebauer not only treated the Babylonian materials in the wrong order, he also failed to provide the necessary groundwork and the requisite tools for the job. Furthermore, commencing with the relatively complex motions of the moon followed by a convoluted treatment of the motions of the two Inferior planets Venus and Mercury was premature in the absence of a coherent treatment of lunar and planetary motion. Which is not to make light of such requirements, even though matters are helped to a considerable extent by the application of essentially the same synodic formulas in both lunar and planetary contexts. Which is hardly surprising in view of the name allotted to the commonly observed period for the moon, *i.e.*, the "synodic" month, simplistically, from one full moon to the next. But what is not simple, however, is that although there is only one moon orbiting Earth, its motions occur at the same time that the latter - while concurrently rotating daily about its axis - is also orbiting the Sun and completing one full revolution of 360° in one year. And once again, there is more than one kind of month and more than one kind of year, not to mention additional inter-related cycles with periodic variations caused by the elliptical natures of the various orbits, inclinations and precession, etc. Plus the fact that the observing platform (Earth) is subject to further variations caused by the considerable tilt of its own rotational axis.

Lastly, it should be noted that so far the present inquiry has been limited to the mean values alone. Omitted so far, the Babylonian treatment of varying velocities encountered in both luni-solar and planetary contexts is far more complex. Again this is *best understood in heliocentric terms*, current negative views on the matter notwithstanding. So, although not wishing to "concatenate without abruption," (as Dr. Johnson was wont put it) it is necessary to now include the Babylonian varying velocities in their various forms plus matters of related interest. But how could this be realized in the face of Neugebauer's bald statement that his analysis of Babylonian astronomy: ³⁷

will reveal to us the working of a theoretical astronomy which operates without any model of a spherical universe, without circular motions and all the other concepts which seemed "a priori" necessary for the investigation of celestial phenomena.

The answer is simple enough. Easily, based on applied Babylonian synodic data and details in the "procedure" texts for Jupiter and Saturn, with both also employing the afore-denied "circular motions," and orbital progress observed, recorded and applied in Babylonian Systems A and B, where further surprises await.

IX. Babylonian Systems A and B synodic arcs and times; elliptical orbits

So far the discussion of the parameters of Systems A and B has been largely concentrated on the apsidal synodic arcs of System A, the mean synodic arcs of System B, an intermediate mean arc from System A' and the unattested theoretical mean values which resulted in Fibonacci ratios of $8/5$, $5/3$, $8/3$, $144/55$, $55/144$ and $144/89$. Yet there are unfortunately difficulties with System B as Aaboe's 1964 analysis of BM 37089 demonstrated which can be traced to a lack of awareness of the underlying formula for the determination of the extremal arcs of System B. But this itself is not the only issue since a secondary part of the problem also involves the role played by the mean synodic month ($MSM = 29;31,50,8,20^d$) in Babylonian planetary theory. It is here, however, that the choice of the slightly too high (but convenient) sidereal year MYR of $12;22,8$ mean synodic months (or $SYR = 365;15,38,17,44,26,40^d$) becomes clear, especially when used with the 360° Zodiac's $12 \times 30^\circ$ zodiacal months and "tithis," *i.e.*, thirtieths of the mean synodic month. In other words, rather than determine the time required for Jupiter to move the mean and varying synodic arcs using the standard year in days and mean synodic months (SYR and MYR above) parts of the work were done using thirtieths, which is a major simplification for a base-60 numerical system. And with pre-selected parameters, also finite, which is where the rounding of the mean synodic arc for Jupiter of $(u) = 33;8,45^\circ$ (from $33;8,44,48,29,..^\circ$) unfortunately also complicates matters, at least for those unfamiliar with the methodology.

The reason for this situation lies in the fact that while Babylonian "procedure" text ACT No. 812 provides a method for the mean synodic time for Jupiter in the badly damaged first section and an alternate approach is given in the second, the two methods yield different answers. Or more precisely, results given to *three* and *five* sexagesimal places respectively - "13;30,27,46" in the first section³⁶ and "13;30,27,46,16,40" deducible from the conclusion of the second. Now *had this value* been given in full in the latter it is safe to say that Neugebauer (who was left "completely in the

dark" by the three-place version in the first section) would almost certainly have realised that both parameters were the mean synodic period of *Jupiter expressed in mean synodic months* with three-place 13;30,27,46 MSM the value applied in the first section.⁵⁷

An unfortunate consequence of the damaged state of Section 1 of ACT 812 is that already condensed details are further obscured, but the mention of an alternate approach in Section 2 suggests that the first method was merely the application of the Babylonian integer period relation for Jupiter, essentially, where MYR = 12;22,8 MSM:

$$36 \text{ revolutions (Z) and } 391 \text{ synodic periods or arcs (II) in } 427 \text{ (N) years} \quad (4)$$

with the time for the mean synodic cycle simply $427/391 \cdot \text{MYR} = 13;30,27,45,52, \dots \text{MSM}$ necessarily rounded up at the fourth sexagesimal place to 13;30,27,46 MSM (398.89078921... days). Plus, by the same methodology and the integer long period relation for Saturn:

$$9 \text{ revolutions (Z) and } 256 \text{ synodic periods or arcs (II) in } 265 \text{ (N) years} \quad (5)$$

the mean synodic time for Saturn is: $265/256 \cdot \text{MSM} = 12;48,13,26,15 \text{ MSM}$ (378.10183198 days) with both methods producing identical answers because no rounding is required for the mean synodic arc of Saturn ($u = 12;39,22,30^\circ$). From this viewpoint the duality is best demonstrated by the Saturn integer period relation, but there is also a more critical matter involving the first section of ACT 812 to be addressed, which is the determination of the extremal or apsidal synodic arcs from the integer period relations. Which, despite the damaged state of Section 1, is readily apparent from what remains. Neugebauer's transcription and commentary of No. 812 Section 1 plus one addition to line 6 is given below. The *15 pa* in line 5 is the location of the mean value (u) at 15 Sagittarius in the Zodiac; *15 zib* and *15 absin* line 6 are the locations of the maximum (M) and minimum (m) arcs/times at 15 Pisces and 15 Virgo.⁵⁸

No. 812

BM 34221 + BM 34299 + BM 35119 + BM 35206 + BM 35445 + BM 45702 (= Sp.327 + Sp.410 + Sp.II,664 + Sp.II,763 + Sp.II,1034 + SH 81-7-6,107)

Contents: Procedure text for Jupiter and Saturn

Arrangement: O/R

Provenance: Babylon [Sp. and SH]

Section 1

Obv. I, beginning destroyed

Transcription

- ¹[.....] .. *lu-DU-ma*
²[.....] 13,30,27,46 a-rá 6,31 DU-*ma*
³[.....] 13,30,27,46 *qablū*(murub₄-u)
⁴[.....] *tašakkan*(gar-an) 1,48 a-rá 6,31 DU-*ma*
⁵[.....] *ēpuš*(dū -]uš) 15 pa 45,14 u₄-meš
⁶[15 zib 50,7,1]5 15 absin 40,20,45 u₄-meš.

Commentary

It is certain that this section concerns System B of the theory of Jupiter. This is shown by the occurrence of the following constants (cf. Introduction p. 311):

Lines 2 and 4	6,31 = II	(= 391 mean synodic arcs/periods, added)
line 4	1;48 = d	of ΔB or ΔT
line 5	45;14' = u	of ΔT
{line 6	50; 7,15' = M	of ΔT (added)}
line 6	40;20,45' = m	of ΔT

The value 50;7,15' = M of ΔT is restored from the parallel passage of No. 813 Rev. I,5. The combination of these two sections shows that it was assumed that m corresponds to 15, M to 15 and consequently u to 15 and 15. In System A the apsidal line goes through 12;30 and 12;30.

Completely in the dark, however, remains the number ... 13,30,27,46 in lines 2 and 3. In line 3 it seems to be called "mean value". The same number occurs in the parallel passage of No. 813 Section 12 but once broken into [13,30] at the end of a line and 27,46 at the beginning of the next line. This may be either a real separation or simply split writing; no other passage shows a separation between 13,30 and 27,46. In No. 813 the number [13,30] 27,46 seems to be added to 45,14 which is the mean value of ΔT.

In line 4 the difference $d = 1;48$ is multiplied by the number period II = 6,31. The result (11,43;48) is not preserved but there is hardly space left for more than the number which would represent the total variation $2(M - m)Z$ of ΔT. In line 2 the "mean value" (?) ... 13,30,27,46 is multiplied by II, again for unknown reasons."

Otto Neugebauer, *Astronomical Cuneiform Texts*, ACT (1955:392-393).

Perhaps the most striking part of Neugebauer's analyses of ACT 812 Sections 1 and 2 is a puzzling inability to come to terms with what must surely be the elementary determination of the mean synodic period for Jupiter *in mean synodic months* followed by the same for the varying periods of System B. Plus, even though Section 1 is damaged, there is still sufficient information to understand precisely what the remaining parameters mean with confirmation found in the "alternate method" described in Section 2, which according to Neugebauer's translation teaches:⁵⁹

... the computation of the difference between synodic time and synodic arc (disregarding the trivial 12 months) following exactly formula (14), Introduction p. 286, for $i = 1$.

[Alternate (method): 33;8,45, the mean(-value) of the longitudes, multiply by 0;1,50,40, and (you obtain) 1;8,8,20. Add to it 11;4, and (you obtain) 12;5, 8, 8,20. Put it down for the gabari's of (one) year.

The first number, 33;8,45', is the mean synodic arc $\Delta\lambda = u$ in Systems B and B'. The epact is $\epsilon = 11;4$ and consequently

$$\frac{\epsilon}{6,0} = 0;1,50,40'.$$

Thus our text follows the formula $u \frac{\epsilon}{6,0} + \epsilon = 12;5,8,20'.$

The abbreviated value 12;5,8' is known from System B' (cf. Introduction p. 311). A still more rounded-off value 12;5,10', is applied in the majority of the ephemerides. For the term gaba-ri cf. below, p. 413.

What follows must mean that the value 12;5,8,20' should be added to the synodic arc. Unfortunately, some of the corresponding signs are damaged, but one may perhaps read

[From (one) appearance to (the next) appearance, (the arc) between them you put down. 12;5,8,20 you add to it and predict the [dates]s.

X. Jupiter

Now had Neugebauer *carried out* the instructions he would have obtained 12;5,8,20 + 33;8,45 = 45;13,53,8,20 as opposed to the *mean* value "45;14' = u of ΔT " with constants for the *maximum and minimum times* also included between lines 5 and 6 of Section 1. So just where was Neugebauer's problem? Apart from temporal racism, in the present case it can be traced to one line in the latter's translation and commentary for Section 2 of ACT 812 where he disregards "the trivial 12 months," except that elsewhere he stipulates that the 12 months should be added to to obtain the synodic times, which is indeed the last requirement to actually predict the dates. But one other earlier step, which is to convert back to days after carrying out key parts of the computation in "tithis" was not apparently carried out. As a result, his final representations of the synodic times for Jupiter using the above values remained in mixed format. Thus 12 mean synodic months of 29;31,50,8,20 days to be added in the final step to values were instead combined by Neugebauer with results which were still in tithis yet still needed to be divided by 30 before the last addition of the 12 months. Therefore division of the end product in the text by 30, *i.e.*, 45;13,53,8,20/30 = 1;30,27,46,16,40 plus the final addition of 12 *mean synodic months* results in a mean synodic period of Jupiter of 13;30,27,46,16,40 months (MSM). Which can then be compared to the three-place number 13;30,27,46 (MSM) in Section 1 - the unknown parameter which left Neugebauer "completely in the dark."

Once aware of the latter result, however, the application of the concept to the contents of lines 5 and 6 of Section 1 naturally follow, *i.e.*, subtraction of the mean synodic arc 33;8,45° (located at Sagittarius 15) from 45;14, the like subtraction of the maximum synodic arc $M = 38;2^\circ$ from 50;7,15 (located at Pisces 15) and a final subtraction of the minimum arc $m = 28;15,30$ from 40;20,45 located at Virgo 15 all result in the more convenient constant 12;5,15.

It turns out that there are four constants: the most accurate being that derived in Section 2 of ACT 812, the next 12;5,8, followed by 12;5,10 and lastly, the above value 12;5,15 as the least accurate but the most convenient to use. In actual fact, the differences for the final periods in days for all four constants are quite small, *e.g.*, using the mean synodic arc (u) for Jupiter of 33;8,45° and MSM (29;31,50,8,20 days), the periods in MSM and days are:

Jupiter k	k + 33;8,45	Divided by 30	+ 12 MSM	Synodic S (days)
12;5,8	45;13,53	1;30,27,46	13;30,27,46	398.890789
12;5,8,8,20	45;13,53,8,20	1;30,27,46,16,40	13;30,27,46,16,40	398.890827
12;5,10	45;13,55	1;30,27,50	13;30,27,50	398.891336
12;5,15	45;14	1;30,28	13;30,28	398.892703

Table 6. Babylonian Jupiter time constants and resulting periods (S) in days.

with the slightly low constant 12;5,8 now explained by its use to generate the "mystery" period 13;30,27,46 MSM.

More critically, however, Neugebauer does not explain that the multiplication factors in Section 2 originate from the relation which incorporates "tithis" (thirtieths of the mean synodic month, t), *i.e.*, $30 \cdot 12;22,8 \text{ MSM} = 371;4$, with division by 360° providing a time per degree for the motion of either Earth (or Sun) of $1;1,50,40$. Which is where the simple addition of the mean synodic arc (u) 33;8,45° then $(u) + N \cdot d$ can be seen as a separate "multiplication" factor of $1x$ from $1;1,50,40$ with the results recorded in a separate column as the first half of the task with the second the corresponding times from the standard constant(s) based on $0;1,50,40$ or $0;1,50$, etc.

XI. Determination of the Babylonian extremal synodic arcs/times and the varying values of System B

Up to a point this problem can be said to have already been dealt with by Neugebauer's detailed descriptions in *THE PLANETARY THEORY IN GENERAL* in *Astronomical Cuneiform Texts* (1955:284-287). Except for two fundamental issues, which is that no instructions were given by Neugebauer to generate a System B from scratch, despite his familiarity with the components necessary for this purpose. This is also why Aaboe (1964:32) was unable to develop a System B from the BM 37089 data for the Earth/Sun based on the difference $d = 0; 0,1,32,42,13,20^\circ$ per day, and although incomplete, the equally extensive daily velocities in the text. The second deep-seated problem is how any progress could be expected in the absence of a fictive approach to planetary motion and/or some form of model.

Returning to this central issue in the context of ACT 812 Section 1 the method for establishing the apsidal synodic arcs becomes apparent from the residual parts of this text, specifically lines 2 and 4 with Neugebauer's comments in part also emphasizing the likely generation of the mean synodic time:³⁹

... Completely in the dark, however, remains the number ...]13,30,27,46 in lines 2 and 3. In line 3 it seems to be called "mean value". The same number occurs in the parallel passage of No. 813 Section 12 ... (where) the number [13.30] 27,46 seems to be added to 45,14 which is the mean value of ΔT ."

In line 4 the difference $d = 1;48$ is multiplied by the number period $II = 6,31$. The result (11,43;48) is not preserved but there is hardly space left for more than the number which would represent the total variation $2(M - m)Z$ of ΔT . In line 2 the "mean value" (?) ...] 13,30,27,46 is multiplied by II , again for unknown reasons."

Recalling again that the Babylonian long integer period relation for Jupiter is:

$$36 \text{ revolutions (Z) and } 391 \text{ synodic periods or arcs (II) in } 427 \text{ (N) years} \quad (4)$$

and now aware that the mystery number "13,30,27,46" is unequivocally the mean synodic period for Jupiter in mean synodic months it is clear that the multiplication of this constant by 391 (the number of synodic periods for Jupiter) yields the total synodic motion over 427 years and 36 revolutions. Whereas the further multiplication of 391 by the difference $d = 1.8$ ($391 \cdot 1;48$) yields a total of 11,43;48 (703.8). The division of the latter by 36 therefore provides the number of increments per synodic arc for one revolution = 19;33 (19.55) with the further division by two giving the amplitude = 9;46,30 (9.775). As in fact included by Neugebauer in his analysis of Babylonian System B for Jupiter where the subtraction of the minimum synodic arc (m) of $28;15,30^\circ$ from the maximum (M) of $38;2^\circ$ results in $9;46,30$.

But this range - *aphelion to perihelion* - can be halved again to take advantage of the relation for the mean synodic arc (u) = $\frac{1}{2}(m + M)$ whereas the parameter P is $(391/36)$ and the full amplitude also $Pd = 19.55$. Therefore, with the inclusion of the mean synodic arc (u) the maximum (M) and minimum (m) synodic arcs for System B can be derived from the general relation:

$$\text{Babylonian System B extremal synodic arcs } (M, m) = u \pm \frac{1}{4}Pd \quad (6)$$

From this viewpoint the period T is dependent on the mean synodic arc (u) with the only variable available d , the amount which increases from aphelion to perihelion and *vice versa* from the latter back to aphelion, etc.

XII. System B locations and times for the extremal/apsidal synodic arcs of Jupiter

Based on the integer period relation (4) above, the parameters of System B for Jupiter reduced to their simplest form are:

11;51,40 (11.86111*) years for the period of revolution T (427/36)
10;51,40 (10.86111*) years for the parameter P ($391/36$, $360/u$) and/or $T - 1$
1;48° for the difference per synodic arc d (variable(?), according to choice)

while relation (6) provides the extremal velocities (m) and (M) which, along with the mean synodic arc (u) below completes the set:

28;15,30° for the minimum synodic arc (m)
33; 8.45° for the mean synodic arc (u)
38;2° for the maximum synodic arc (M)

Corresponding times - as shown in Table 14 - are dependent on the choice of constant (i.e., 12; 5,8 through 12;5,15).

However, System B' used in ephemeris ACT No. 640 and procedure text ACT No. 813 Sections 21 and 22⁶⁰ utilize a value for d of 1;46,40 instead of the standard value 1;48 which, with u and P remaining constant, results in new values for M and m based on $\frac{1}{4}Pd = 4;49,37,46,40$ with extremal velocities of $28;19,7,13,20^\circ$ and $37;58,22,46,40^\circ$. All three may be compared to the equivalents given by Neugebauer of $9;39,10^\circ$ (the half = $4;49,35$ is not included by the latter) followed by $28;19,10^\circ$ and $37;58,20^\circ$ deduced from the above texts. Neugebauer's treatment of this variant includes both a change in the underlying period while still maintaining the standard mean synodic arc of $33;8,45^\circ$. It also contains "certain discrepancies ... which are not exclusively due to rounding-off for the sake of easier computation." These include the observation that replacement of the constant 12;5,8,8,20 for the mean synodic

time with the lesser value 12;5,8 is “a permissible approximation.” And certainly, rounding applied to the four-place data to reduce it to two-places is indeed reasonable. Yet there remains a problem since Neugebauer’s constants for the positions and times also vary. This is no small matter, although it does serve to emphasize that the methodology, - despite numerous variants and details - was not fully understood, especially when *tithis and the mean synodic month* are involved, as in the present case.

More to the point, Neugebauer also provides the following commentary concerning a “third method” supplied in Sections 14,15 and 16 of ACT 813 for System A’ oddly exemplified by the use of much rounded parameters for the velocity of the “Sun.” Thus he writes:

In spite of many difficulties in the reading of these sections, which are headed ‘third (method)’, the main contents can be established. We are dealing with the slow, medium, and fast arcs of System A’. The synodic arcs 30 and 33;45 are given, and the corresponding synodic times are to be found. In order to travel one degree the mean sun requires $\frac{1;0,57,13}{0;59,8} = 1,50,49, \dots$. If we abbreviate this value we obtain for a synodic arc of w° a time of $w + 0;1,50w$.

We know furthermore that 12 lunar months are 11;3,20 shorter than one solar year (lines 10, 11, 15). Thus the time difference in question is found in Section 15 as follows: (added: zodiacal names included for the signs)

From 9 Cancer (*sic!* for Virgo) to 2 Capricorn (from) year to year 33;45. . . (from) year to year 33;4[5 . . .] multiply by 0;1,50 and (the result) 1;1,52,30 add to 33;45. 34;46,52,[30] and 11;3,20, the days of the sun, add together and 45;50,[12.30 put down for the gaba-ri].

In Section 16 the factor 0;1,50 appears in the form 0;2 - 0;0,10:

From 2 Capricorn to 17 Taurus (from) year to year 36 . . . [. . .] . . . [.] 36 . . . multiply by 0;2 and [(the result) 1;12 36 multiply by 0;0,10 and] subtract 0;6 from 1;12 and what remains (namely) 1;6 [add to 36 and to 11;3,20. The result is 48;9,20

In Section 14 one should expect for the slow arc a time difference of $30 \cdot 1;1,50 = 30;55$ whereas the text gives only 30;45:

From 9 Cancer to 9 Scorpio] 30 (degrees or 1) danna and (from) year to year 11;3,20 days subtracted(?) and (?) 30 (degrees or 1) dan[na] . . . of Jupiter (from) year to year . . . 30;45 days multiply(?) and 30;45 days [and 11;3,20] days add together and 41;28,20 days put down for the gaba-ri of (one) year.

Neugebauer, *Astronomical Cuneiform Texts*, ACT (1955:392-412).

Before discussing the above and the accompanying implications, there is, however a more immediate issue which concerns Neugebauer’s introduction above of the ratio “1;0,57,13 / 0;59,8” and an approximate result of “1,50,39...” This is a mystery, since in Section 13 just before (which is parallel to Section 2), his translation included 0;1,50,40 *en route* to the determination of the constants for 33;8,45 (Jupiter (*u*), *i.e.*, intermediate values to be added (1;1,8,8,20 and 11;4) to obtain the final value 12;5,8,8,20. This is followed (as in the parallel text in Section 2) with condensed instructions for its use, which are worth repeating:

Alternate (method): 33;8,45, the mean(value) of the longitudes, multiply by 0;1,50,40, and (you obtain) 12;8,8,20]. Add to it 11;4, and (you obtain) 12;5, 8, 8,20. Put it down for the gaba-ri of (one) year. The arc [between them you put down. 12;5,8,8,20 add to it and predict the dates. 1,1,8,820 . .]

The reason behind labouring this point is that there seems to have been insufficient understanding of the central role played by the inter-related trio of luni-solar constants applied in Babylonian astronomy, namely, with assigned abbreviations introduced earlier, the following:

1. Mean Synodic Month (MSM) = 29;31,50,8,20 days (29.530594135802474)
2. Sidereal Year in MSM (MYR) = 12;22,8 Mean Synodic Months (12.36888*)
3. Sidereal Year in Days (SYR) = SYR = 365;15,38,17;44,26,40 days (365.260637688614546)

to which can be added the Zodiacal “Year” of 360° with its $12 \cdot 30^\circ$ months of the Zodiac and the associated relations:

$$\begin{aligned} \text{MSM} \cdot \text{MYR} &= \text{SYR} \quad (\text{MYR} = 12;22,8 \text{ MSM}) \\ 30 \cdot \text{MYR} &= 371;4 \quad (6,11;4 = 371.10666^* \text{ tithi}) \end{aligned}$$

Thus arriving at the multiplication factor for the conversion *to tithi per degree* of 1;1,50,40 as used in the texts, along with a neat and simple method for concurrently recording the varying synodic arcs and their corresponding times.

$$\frac{30 \cdot \text{MYR}}{360} = \frac{\text{MYR}}{12} = \frac{6,11;4}{6,0} = 1;1,50,40 \quad (1.03074074074^*)$$

Which is to *bifurcate* the multiplication factor 1;1,50,40, i.e, first multiply the synodic arcs by the integer 1, then set this aside (or record) the “result” as the arc under consideration for subsequent positional assignment(s) in degrees, remembering that the conversion to tithis will later require conversion to days. Meanwhile, with unity (1) removed the remainder of the multiplication factor (0;1,50,40) can be applied to the arc itself. Thus essentially an operation in five stages:

- (1) Synodic arc • 1;1,50,40 for the arc with or without the difference (s) *d*.
- (2) Syndic arc • 0;1,50,40 + *k* = 11;4 (or 11;3,20)
- (3) Step (1) combined with Step 2.
- (4) (Step (3) sum divided by 30 to convert to MSM
- (5) 12 MSM added to Step 4 for the corresponding Time in MSM

with, for the second part, the simpler two-place constant 0;1,50 as used in ACT 813, Sections 14 to 16.

Except that instead of MYR = 12;22,8 MSM the length of the year based on the replacement of 11;4 by 11;3,20 is 12;22,6,40 (12.36851851851852 MSM), the total in tithis is 371;3,20 (371.0555*) and the multiplication factor now 1;1,50,33,20. Which further explains why the later was reduced to 01;1,50 while also retaining the 1X factor for the for the synodic arc.

XIII. Sections 14, 15 and 16 of ACT 813 revisited

The major point of interest is that the 30°, 33;75° and 36° synodic arcs of System A' examined earlier with respect to the *Fibonacci* ratios found in the variants of relations 7, 8 and 12 are here assigned specified sectors in the Zodiac:

- 30° Minimum arc (*m*) assignment: 9 Cancer ☉ to 9 Scorpio ♏.
- 33;45° intermediate arc (*u₂*) assignment: 9 Scorpio ♏ to 2 Capricorn ♐
- 36° Maximum arc (*M*) assignment: 2 Capricorn ♐ to 17 Taurus ♉

with a line of apsides from 12;30 Virgo ♍ to 12;30 Pisces ♈ according to Neugebauer and Act 814, Section 2.⁶¹

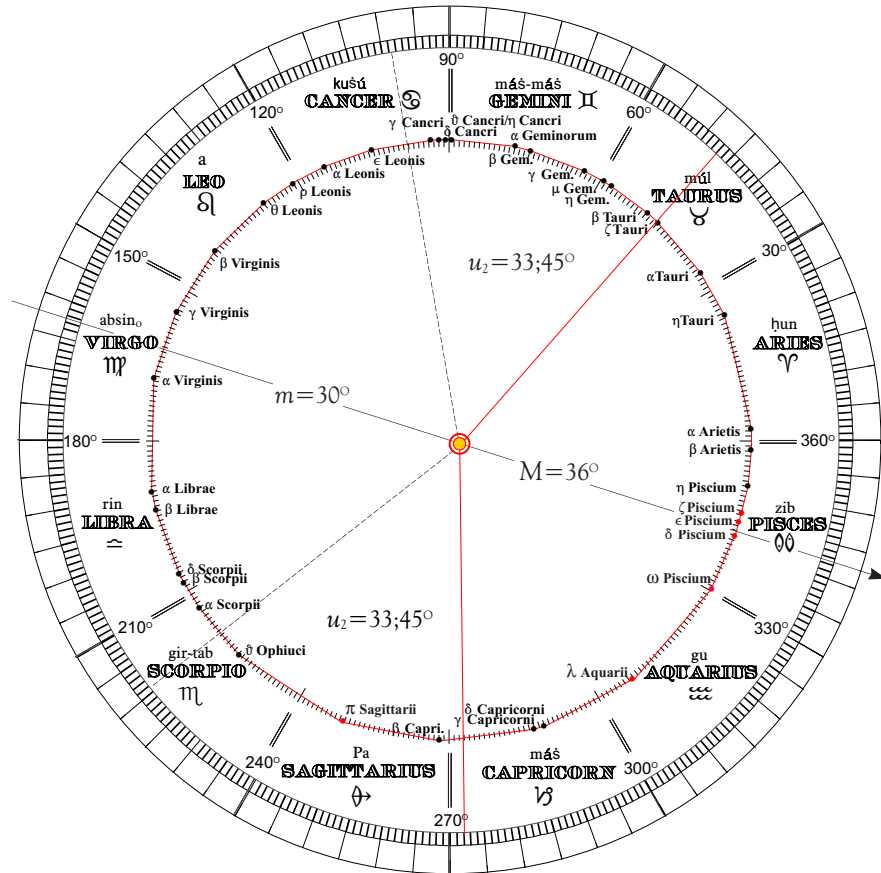


Fig 8. Jupiter System A' assigned sectors for the 30°, 33;45° and 36° synodic arcs

At which point it seems necessary to investigate this matter further in terms of elliptical orbits in general in light of the above assignments and previous discussions concerning the Babylonian synodic arcs.

Before returning to the ellipse in Babylonian contexts mentioned earlier, it may be helpful for the casual reader to revisit Kepler's three laws of planetary motion described and qualified by Cesare Emiliani (1995) as follows:⁶²

FIRST LAW: 'The orbit of a planet is an ellipse with the Sun at one of the two foci.'
This law establishes the shape of planetary orbits. They are not circles, but ellipses.

SECOND LAW: 'A planet revolves around the Sun with the connecting lines sweeping equal areas in equal times.'

This law establishes that the planets move faster when they are closer to the Sun and more slowly when they are farther away.

THIRD LAW: 'The square of the sidereal period of a planet is proportional to the cube of the semimajor axis of it's orbit.'

This law establishes that the sidereal period of the planets (the time it takes to go around the Sun once) increases as the distance from the Sun increases."

Cesare Emiliani, *THE SCIENTIFIC COMPANION*, (1995:121)

For ellipses in the present context modern textbooks generally provide the mean distances *and* eccentricities in *a priori* formulas for the calculation of apsidal data, but what about the earlier Babylonian methodology ? As it so happens an alternative approach to the subject is provided in Robin S. Green's description of "*Kepler's equation for a bound orbit*," published in *Spherical Astronomy* (1985), the relevant parts of which are as follows:⁶³

"... $F(r)$ is simply shorthand for the quadratic form given by

$$F(r) = r^2 - 2ar + h^2 a/u. \quad (6:19)$$

Equation (6:18) implies that $F(r)$ is ≤ 0 . So we may conclude that the radius vector r is restricted to the range

$$r_1 \leq r \leq r_2. \quad (6:20)$$

where r_1 and r_2 are the two roots of $F(r) = 0$. In fact, *they must correspond to the perihelion and aphelion distances*.

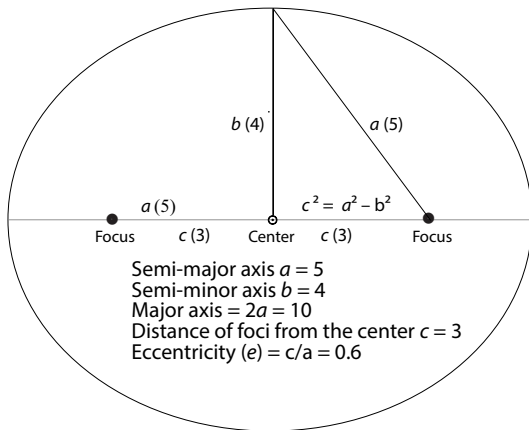
It should be noted, from equation (6:19), that $r_1 + r_2 = 2a$. Moreover, it is convenient to introduce the parameter e - later to be identified with the orbital eccentricity - as

$$e = \frac{r_2 - r_1}{r_2 + r_1} \quad (6:21)$$

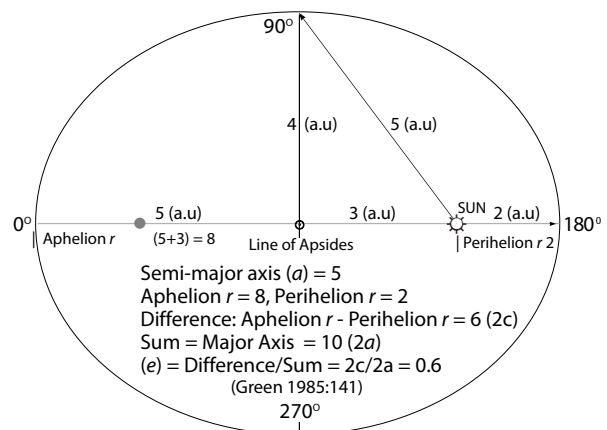
Then, working in terms of a and e , it is seen that $r_1 = a(1 - e)$, $r_2 = a(1 + e)$ (6:22)

Thus in the present context, all that will be required is the conversion of the Babylonian extremal/apsidal synodic arcs from degrees to time (T), then (via the harmonic law) to distance (R) to conform with relations (6:20) and (6:21) for the eccentricity and the remaining parameters.

Finally, in terms of background information it may prove beneficial to re-evaluate Green's relation (6:21) applied to the well-known ellipse based on the 5-4-3 Pythagorean triple in both standard and astronomical contexts:



I. Ellipse based on the 5-4-3 Pythagorean triangle



II. The 5-4-3 ellipse in astronomical context

Fig 9. I, II. The ellipse based on the 5-4-3 Pythagorean set in both general and astronomical contexts

Here the integral relationship between the semi-major axis 5 (a), the semi-minor axis 4 (b) and 3, the parameter (c) is routinely expressed as $a^2 = b^2 + c^2$ with the major axis $2a = 10$, the minor axis $2b = 8$, the distance between the two foci $= 2c$ and eccentricity (e) from $c/a = 0.6$.

XIV. Apsidal distances for the elliptical orbit of Jupiter

System A maximum synodic arc (W) of 36° and minimum synodic arc (w) of 30° of Jupiter can be applied to the following variant of relation (13): $P = 360^\circ/u = T - 1, T = P + 1$ (13)

i.e., from $T = (P + 1)$ and :- $\text{Heliocentric distances } R = [(360^\circ/\text{Synodic Arcs } W, w) + 1]^{2/3}$ (13T)

to generate (via the harmonic law or variant) the following apsidal data for Jupiter with reference to unity ("a.u.).

Perihelion distance R from $(360^\circ/36^\circ) + 1, T = 11, R = 11^{2/3} = 4.9460874$ (a.u.)
 Aphelion distance R from $(360^\circ/30^\circ) + 1, T = 13, R = 13^{2/3} = 5.2374311$ (a.u.)
 $(e) = (\text{Aph } R - \text{Perh } R)/(\text{Aph } R + \text{Perh } R) = 0.05562721$ (modern: 0.04849485)
 (determination of eccentricity e , R. M. Green, *Spherical Astronomy* 1985:141)⁶¹

The same process applied to System B extremal synodic arcs for both Jupiter and Saturn results in lower correlation for the eccentricities, whereas System A extremal synodic arcs for Saturn prove to be the most useful.

XV. Apsidal distances for the elliptical orbit of Saturn

Saturn's System A apsidal synodic arcs generate corresponding *heliocentric distances* from the maximum (W) synodic arc $14;3,45^\circ$ and (w), the minimum synodic arc $11;43,7,30^\circ$ as follows, with rounded distances included for a more immediate Babylonian application:

Perihelion R from $(360/W)+1, T = 26.60, R = T^{2/3} = 8.9108902$ (a.u.), rounded: "9"
 Aphelion R from $(360/w)+1, T = 31.72, R = T^{2/3} = 10.0204861$ (a.u.), rounded: "10"
 Mean Distance $(265/9, \text{Mean period}, T, R = T^{2/3} = 9.5353267$ (a.u.) rounded: "9.5"
 Distance between foci (d) = 1.109595829, $c = 0.5547979145$, rounded: "1" & "1/2"
 $(e = 0.0586115; \text{rounded } e = 1/19 = 0.0526316. \text{ modern } e = 0.0555086)$

All of which come into finer focus when applied to a Babylonian mathematical cuneiform text of hitherto unknown practical significance. As described by Marcus du Sautoy,⁶⁴ the text involves a procedure known as "completing the square," which - as du Sautoy notes - is equivalent to solving the quadratic equation $x^2 - x = 870$.

The latter's description of the text is given next in decimals for clarity with apsidal distances for Uranus similarly assigned to the right of the original text and the rounded data for Saturn applied next as the working example.

I have subtracted the side of my square from the area: 870.
 You write down 1, the coefficient.
 You must break off half of 1.
 0.5 and 0.5 you multiply. You add 0.25 to 870. Result 870.25.
 This is the square of side 29.5.
 You add 0.5, which you multiplied, to 29.5.
 Result 30, the side of the square. [ORIGINAL text]
 [Quadratic equation : $x^2 - x = 870$]

I have subtracted the side of my square from the area: 380.
 You write down 1, the coefficient.
 You must break off half of 1.
 0.5 and 0.5 you multiply. You add 0.25 to 380. Result 380.25.
 This is the square of side 19.5.
 You add 0.5, which you multiplied, to 19.5.
 Result 20, the side of the square. [URANUS (added)]
 [Quadratic equation : $x^2 - x = 380$]

The original Babylonian Mathematical Problem

I have subtracted the side of my square from the area: 870.
 You write down 1, the coefficient.
 You must break off half of 1.
 0.5 and 0.5 you multiply.
 You add 0.25 to 870. Result: 870.25
 This is the square of side 29.5.
 You add 0.5, which you multiplied, to 29.5
 Result 30, the side of the square."

Method applied to the heliocentric elliptical orbit of Saturn based on rounded Babylonian System A data.

(Added : Quadratic equation : $x^2 - x = 90$; eccentricity $e = 1/19$)

I have subtracted the side of my square from the area: $90 = b^2$
 You write down 1, the coefficient. $d^* = 1$
 You must break off half of 1. $\frac{1}{2}d = c = 0.5$
 0.5 and 0.5 you multiply. $c^2 = 0.25$
 You add 0.25 to 90. Result: 90.25 $c^2 + b^2 = 90.25$
 This is the square of side 9.5. From $a = \sqrt{(b^2 + c^2)}$
 You add 0.5, which you multiplied, to 9.5 $(c+a) = 10$, aphelion r
 Result 10, the side of the square. Saturn, aphelion r
 also, (added) Subtract 0.5 from 9.5 = 9 Saturn, perihelion r
 Saturn, mean distance : $R = 9.5$ (relative to unity)

* d = difference between aphelion and perihelion distances

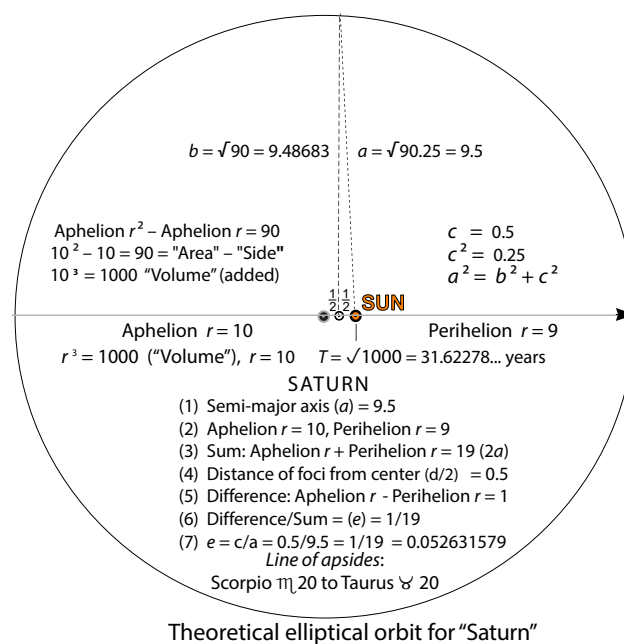


Fig 10. The Babylonian mathematical problem and elliptical orbit of Saturn from rounded Babylonian System A data.

Similarly applied, with fixed "coefficients," $d = 1$, $c = 0.5$, the data for Uranus and Neptune are therefore.

Uranus: ($b^2 = 380$), add 0.5 to 19.5 ($c + a$) = 20, *aphelion r*
(Added: Quadratic equation : $x^2 - x = 380$; eccentricity $e = 1/39$)
(Added), subtract 0.5 from 19.5 ($a - c$) = 19, *perihelion r*
Uranus, mean distance : **R = 19.5** (relative to unity)

Neptune: ($b^2 = 870$), add 0.5 to 29.5 ($c + a$) = 30, *aphelion r*
(Quadratic equation : $x^2 - x = 870$, the original problem; $e = 1/59$)
(Added), subtract 0.5 from 29.5 ($a - c$) = 29, *Perihelion r*
Neptune, mean distance : **R = 29.5** (relative to unity)

XVI. Uranus and Neptune

The two theoretical positions exterior to Saturn require that the "coefficient" d (1) and its half c remain constant, thus the eccentricities diminish with distance commencing with "sides of squares" of 10 (as used), 20 (unattested, but sequential), and 30, again sequential and the original target. In other words, the aphelion distances of 20 for Uranus ($e = 1/29$) and 30 a.u. for Neptune ($e = 1/59$). Lastly, on a more practical note, unaided detection of Uranus cannot be ruled out in any case; the latter, although faint is undoubtedly visible to the naked eye.⁶⁵⁻⁶⁷ Alternatively, if the 2-(3)-5 Fibonacci series is theoretically assigned to the periods of Saturn, Jupiter-Saturn synodic SD1 and Jupiter, then the series will continue outward for two more planetary positions before ultimately ending at unity.

Thus commencing with the period of Neptune (1), Uranus-Neptune synodic also 1 and the period of Uranus 2 the sequence is followed next by 1-(2)-3 to include the latter. Continuing inwards with the resonant Saturn-Jupiter triple [2-(3)-5] the sequence then proceeds in due order with 3-(5)-8, 5-(8)-13, 8-(13)-21, 13-(21)-34 and 21-(34)-55, etc. Thus essentially the inclusion of the intermediate synodic cycles to complete the Pierce-Agassiz model which finally became the extended *Phi*-series planetary framework.

Moreover, though wandering far afield, the above requirement that the "coefficient" d remains unity (1) suggests that certain "Pythagorean" matters involving the number 216 in Vitruvius' *Ten Books of Architecture* might have some bearing on the discussion. In particular, what may best be described as a "columnar" analogy, where the following three architectural "styles" are grouped together, as are the mean distances of **9.5**, **19.5** and **29.5** for the Saturn, Uranus and Neptune ellipses assigned above, with the "coefficient" ($d = 1$) and half ($c = 1/2$) incorporated thus:⁶⁸

In the *systyle*, let the *height* be divided into *nine and a half parts*, and *one of these given to the thickness of the column*. If the building is to be *systyle* and monotriglyphic, let the front of the temple if *tetrastyle*, be divided into *nineteen and a half parts*; if *hexastyle*, into *twenty-nine and a half parts*. (emphases supplied)

VITRUVIUS: *The Ten Books on Architecture*, Bk III, Ch. III, 10. (trans. M H. Morgan, 1960:84)

XVII. Completing the Square

The application of rounded System A parameters for Saturn permits a comparable solution to the problem and also supplies a practical meaning which can be expanded to include *theoretical distances* for both Neptune and Uranus. This does not, of course, establish that either of the orbits of the outermost planets were known *per se*, but rather, that the suggested data can be interpreted as *theoretical extensions* beyond Saturn, the outermost planet known in Antiquity. As for "completing the square, this is readily demonstrated by dual figures for the Saturn data:

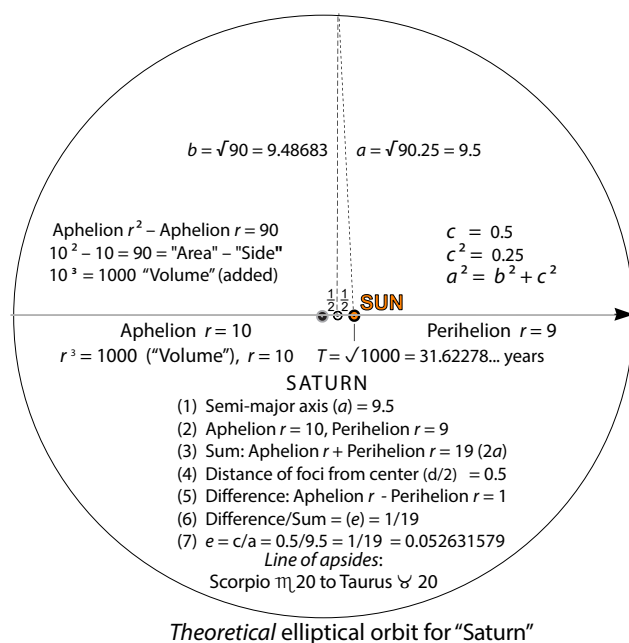


Fig. 11a. Saturn ellipse with rounded System A distances.

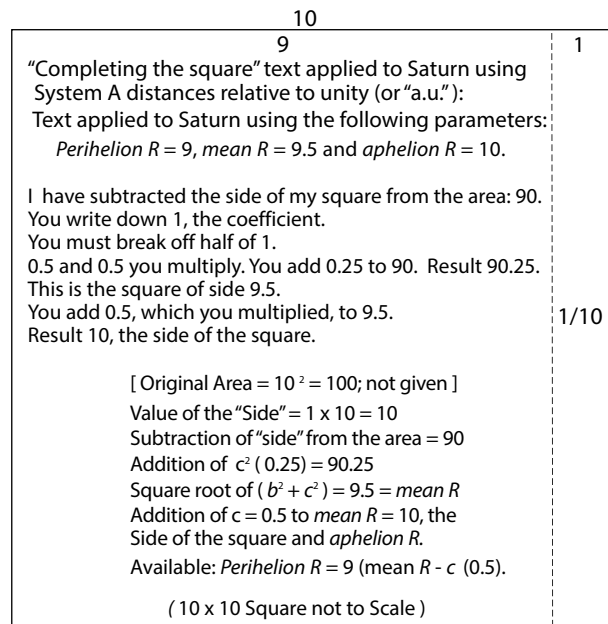


Fig. 11b. Completing the square procedure applied to Saturn.

Thus *aphelion distances* of 20 (a.u.) for Uranus and 30 (a.u.) for Neptune with the latter's mean distance of 29.5 a.u. providing a B1 base period of 160.2260124 years and 164.3167673 years from the solution to the problem ("30"). All of which is sensibly heliocentric in both form and application with the mean heliocentric distances known and aphelion distances the target. Which also makes perfect sense, since the mean and aphelion distances are all that are required to supply the remaining parameters for practical ellipses with assigned lines of apsides. Therefore, if indeed astronomical in the above sense, then prior tasks from a modern viewpoint would entail the calculation of *cube roots* from the *squares of the periods of revolution* (T), thus "Kepler's" two-parameter law of planetary motion: $R = T^{2/3}$. As it turns out, the method used was identical in so much as the final product was still $R = T^{2/3}$ but includes the corresponding velocity with respect to unity (V_r) and inverse (V_i) exemplified by the Triple interval $[3^0, 3^1, 3^2, 3^3] = [1, 3, 9, 27]$. But before continuing in this vein the introduction of orbital velocities requires further expansion.

XVIII. Galileo, Plato, and orbital velocity in earlier times

Although including the time of Plato and latter's *Dialogues* some 400 years before the current era there is another matter which needs to be considered, which is orbital velocity in earlier times. Specifically, velocity expansions for the laws of planetary motion published in the *Journal of the Royal Astronomical Society of Canada* in a 1989 paper by the present scribe ("Projectiles, Parabolas and Velocity Expansions of the Laws of planetary motion") based on Galileo's treatment of orbital velocity in his *Dialogues Concerning Two New Sciences* (1638).⁶⁹ Predicated on Plato's *Timaeus* and *Epinomis* readers are invited to confirm Galileo's reasoning, when, after explaining his understandable hesitancy concerning open discussion of the subject, he states:⁷⁰

"but if anyone desires such information he can obtain it for himself from the theory set forth in the present treatment."

Which is how the present scribe - little more than a tertiary restorer - came to be involved in this matter. But either way the final products were assuredly an improvement over the two-parameter period (T) and distance (R) format currently in use. In other words, instead of the Harmonic or Kepler's Third law of planetary motion: $R^3 = T^2$, a series of additional formulas were now available incorporating *orbital velocity* V_r (relative to unity) and *inverse Velocity* (V_i). Thus the following velocity relations for the laws of planetary motion with related notes as published in 1989:⁶⁸

$$R = V_i^2 \quad (1)$$

$$V_r = R/T, \quad (2)$$

while the further relationships

$$V_i = T/R, \quad (3)$$

$$T = V_i^3, \quad (4)$$

$$V_i^6 = R^3 = T^2, \quad (5)$$

where T is the sidereal period in years, follow from Kepler's Third Law of planetary motion. Relation (5) may also be expressed in exponents (i.e., $[V_i^0, V_i^1, V_i^2, V_i^3]$) and applied to the parabola as *the first three integer sets* which illustrate the Third Law.⁷

NOTES 5 and 7

5. Galileo discusses the sets [1,2,4,8] and [1,3,9,27] with respect to *squaring and cubing* in the *New Sciences*, (First Day [83]). The same sets are also mentioned by Plato in the *Timaeus* (35b and 43d), and the first set [1,2,4,8] is discussed again in *Epinomis* (991a-992a). Added, 2022: [1,1,1,1] (Earth), [1,2,4,8] (no body), [1,3,9,27] (Saturn, *perihelion* V_i, R, T)

7. The ancient relationship between a point, a line, an area, and a volume. See Galileo's discussion of the latter pair and the "sesquialteral ratio" between them in the *Two New Sciences*, First Day, (134-135)."

The paper also included a comparison of the mean velocities of the planets utilizing additional ratios and also a constant velocity for Earth in kilometers per second to prove the value of the approach in practical terms.

Planet	Distance (R)	Period (T)	V_r (R/T)	V_a (kR/T)	Modern
Mercury	0.387107	0.24085	1.60725	47.88	47.89
Venus	0.723350	0.61521	1.17577	35.03	35.03
Earth	1	1	1	29.79	29.79
Mars	1.523733	1.88089	0.81011	24.13	24.13
Jupiter	5.201287	11.86223	0.43847	13.06	13.06
Saturn	9.538188	29.45770	0.32379	9.65	9.64
Uranus	19.182303	84.01390	0.22832	6.80	6.81
Neptune	30.057937	164.79300	0.18240	5.43	5.43
Pluto	39.440188	247.69000	0.15923	4.74	4.74

* **Table 1**, "Velocity Expansions of the Laws of Planetary Motion," *JRASC*, Vol. 83, No. 3, p. 211, June 1989. Reproduced with permission of the Editor.

XV. The Double and Triple Intervals revisited;

The **Double** and **Triple** Intervals: [**1, 2, 4, 8**] and [**1, 3, 9, 27**] are the first pair of integers which demonstrate the *Harmonic Law* relative to unity: $R^3 = T^2$ (4 a.u.³ = 8 years² = 64, and 9 a.u.³ = 27 years² = 729). Plus, in the same sense, the "Single" interval [**1, 1, 1, 1**] assigned to Earth (1 a.u.³ = 1 year² = 1) with the second parameter also providing the names for the intervals, but otherwise (in modern times) of unknown significance.

Surprisingly (and in whatever Babylonian context), the determination of distances relative to unity is nonetheless impressive in so much as it incorporates relative orbital velocity (V_r) and inverse (V_i) to streamline the process and simplify calculation. This much can be surmised from the Babylonian texts under discussion, augmented by the Pythagorean *Tetractys*, Plato's Double and Triple Interval plus Old Babylonian mathematical text VAT 8457 for the eventual calculation of cube roots from the periods of revolution (T).

Even so, there is one additional complication that arises from the simplicity of the final product which is that the corresponding heliocentric distance (R) is *not obtained from the cube root of the period squared* ($R = \sqrt[3]{T^2}$) but the *cube root of the period* (T) itself as the initial goal, e.g., the *cube root* of 27 years is 3, which when squared yields the required distance (R) = 9. *What is the astronomical significance of the cube root of the period of revolution?*

Simply stated and denoted by V_i , this value is *the inverse orbital velocity with respect to unity* corresponding to the *square root of the heliocentric distance R* and the *cube root of the period of revolution T* . Which together, expressed in a format which extends beyond the two-parameter limitations of "Kepler's" Third Law ($R^3 = T^2$) is:

$$\text{Velocity } V_i^6 = \text{Distance } R^3 = \text{Revolution } T^2 \quad (\text{relation 5, 1989})$$

$$\text{Distance } R = \text{Velocity } V_i^2 \quad (\text{relation 1, 1989})$$

with possible origins, it is suggested, demonstrated in Old Babylonian mathematical text VAT 8547.

XVI. VAT 8547 and the extraction of cube roots from periods of revolution

The present approach to the extraction of cube roots in VAT 8547 and similar texts differs from previous treatments which were handicapped by lack of context, whereas it is suggested here that they pertain to astronomy in general and *the cube roots of the periods* in particular. Which, as it so happens, *are also the square roots of the distances*, and therefore together they provide the basis (along with unity) for the "interval" concept, tetradic exponents, "Point Line-Square-Cube analogy" and "completing the cube" methodology. Even so, this does not explain the oddities in VAT 8547 discussed by Neugebauer and Sachs (1945),⁷¹ Sachs (1952)⁷² and Muroi (1989).⁷³ Nonetheless, what follows next is gained from these various commentaries, all linguistic complexities notwithstanding. Matters are, however simplified in so much as the same procedure is applied to all four numbers (or years), i.e., 27, 64, 125 and 216 with the procedure for the cube root of the first number (27) translated by Kazuo Muroi as follows:⁷³

- 1 What is the cube root of 27?
- 2 (When) you (perform the calculation), subtract 0;7,30 from 27 and
- 3 you will leave 26;52,30. The 0;7,30 which you subtracted
- 4 place below 26;52,30 and
- 5 26;52,30 0;7,30. What is the cube root of 0;7,30? 0;30.
- 6 Make the reciprocal 0;7,30 and (you get) 8.
- 7 Multiply 8 by 26;52,30<+>0;7,30 (and you get) 3,36.
- 8 What is the cube root of 3,36? 6.
- 9 Multiply 6 by 0;30, the cube root (and you get) 3.
- 10 3 is the cube root of 27.

which, with decimals substituted for sexagesimal numbers and fractions added, is:

- 1 What is the cube root of 27?
- 2 (When) you (perform the calculation), subtract 0.125 (1/8) from 27 and
- 3 you will leave 26.875. The 0.125 (1/8) which you subtracted
- 4 place below 26.875 and
- 5 26.875 0.125 (1/8). What is the cube root of 0.125 (1/8)? 0.5 (1/2).
- 6 Make the reciprocal 0.125 (1/8) and (you get) 8.
- 7 Multiply 8 by 26.875<+>0.125 (and you get) 216.
- 8 What is the cube root of 216? 6.
- 9 Multiply 6 by 0.5 (1/2), the cube root (and you get) 3.
- 10 3 is the cube root of 27.

The instructions in lines 5 and 7 have caused the most difficulty; but the latter ($26.875/0.125 = 215$) is followed by 216, *i.e.*, with 1 added to “complete the cube.” The procedure uses four inter-related constants applied to 27, 64, 125 and 216 in VAT 8547 to obtain 3, 4, 5 and 6 for the cube roots with (+1) added in Step 3 as follows:

1. The fraction $(1/8) = 0.125$ to be subtracted from the number whose cube root is to be determined.
2. The integer 8 as a constant multiplier.
3. Unity $(1) = 8 \cdot (1/8)$ to be added (*i.e.*, recombined) to “complete the cube” for which an integer cube root is known.
4. A fixed divisor $0.5 (1/2)$ is applied to produce the final cube root of the period with the understanding that the cube root of $1/8$ is $1/2$.

Therefore, for the Ternary, Quaternary, Quinary and Senary “periods” in VAT 8547 of **27, 64, 125** and **216** (years), the cube roots are **3, 4, 5** and **6** obtained in the following consistent manner which (for completeness and to connect with the procedure outlined above) includes the Single [1, 1, 1, 1] and Double interval [1, 2, 4, 8] given below:

SINGLE INTERVAL (UNITY): [1, 1, 1 a.u., 1 year]

1. Subtraction of 0.125 from 1 year $(1 - 0.125) = 0.875$.
2. Multiplication of 0.875 by 8 = 7.
3. Understanding that $(7 + 1) = 8$, a number with a known integer cube root (2)
4. Reduction of the known integer cube root 2 by $1/2$ for the cube root of **1 = 1**.

DOUBLE INTERVAL (BINARY): [1, 2, 4 a.u., 8 years]

1. Subtraction of 0.125 from 8 years $(8 - 0.125) = 7.875$.
2. Multiplication of 7.875 by 8 = 63.
3. Understanding that $(63 + 1) = 64$, a number with a known integer cube root (4)
4. Reduction of the known integer cube root 4 by $1/2$ for the cube root of **8 = 2**.

followed by the cube roots for the four numbers in VAT 8547 (**27, 64, 125, 216 years**) :

TRIPLE (TERNARY) INTERVAL: [1, 3, 9 a.u., 27 years]

1. Subtraction of 0.125 $(1/8)$ from 27 years $(27 - 0.125) = 26.875$.
2. Multiplication of 26.875 by 8 = 215.
3. Understanding that $(215 + 1) = 216$, a number with a known integer cube root (6).
4. Reduction of the known integer cube root 6 by $1/2$ for the cube root of **27 = 3**.

QUADRUPLE (QUATERNARY) INTERVAL: [1, 4, 16 a.u., 64 years]

1. Subtraction of 0.125 $(1/8)$ from 64 years $(64 - 0.125) = 63.875$.
2. Multiplication of 63.875 by 8 = 511.
3. Understanding that $(511 + 1) = 512$, a number with a known integer cube root (8).
4. Reduction of the known integer cube root 8 by $1/2$ for the cube root of **64 = 4**.

QUINTUPLE (QUINARY) INTERVAL: [1, 5, 25 a.u., 125 years]

1. Subtraction of 0.125 $(1/8)$ from 125 years $(125 - 0.125) = 124.875$.
2. Multiplication of 124.875 by 8 = 999.
3. Understanding that $(999 + 1) = 1000$, a number with a known integer cube root (10).
4. Reduction of the known integer cube root 10 by $1/2$ for the cube root of **125 = 5**.

SEXTUPLE (SENARY) INTERVAL: [1, 6, 36 a.u., 216 years]

1. Subtraction of 0.125 from 216 years $(216 - 0.125) = 215.875$.
2. Multiplication of 215.875 by 8 = 1727.
3. Understanding that $(1727 + 1) = 1728$, a number with a known integer cube root (12)
4. Reduction of the known integer cube root 12 by $1/2$ for the cube root of **216 = 6**.

XVII. “Completing the cube” in astronomical context

Just how this connects with a “Point-Line-Square-Cube” analogy becomes apparent *if* the first number in VAT 8547 is the *Triple Interval's* 27-year period of revolution and the remaining periods in this text *the sequential Fourth (64), Fifth (125) and Sixth Interval (216)* in astronomical context.

1 Point ⁰	<i>V</i> _i Line ¹	<i>R</i> Square ²	<i>T</i> Cube ³	[Unity, Velocity, Distance, Period] <i>V</i> _i supplies the names for the Intervals	
1	3	9	27	Triple	Interval (Ternary).
1	4	16	64	Quadruple	Interval (Quaternary)
1	5	25	125	Quintuple	Interval (Quinary)
1	6	36	216	Sextuple	Interval (Senary)

Table 7a. The Four Intervals in VAT 8547 in astronomical context.

Plus in addition, the inclusion of the Single [1,1,1,1] and Double [1, 2, 4, 8] intervals serves to emphasize that the method extends from unity for all integers, while in passing providing (via the distances) the dimensions 1 x 4 x 9, i.e., those of Sir Arthur C. Clarke's enigmatic, fictional 2001 *monolith*...

1 Point ⁰	Vi Line ¹	R Square ²	T Cube ³	[Unity, Velocity, Distance, Period] Vi supplies the names for the Intervals	
1	1	1	1	Single	Interval, ref. Vi, R, T.
1	2	4	8	Double	Interval (Binary)
1	3	9	27	Triple	Interval (Ternary)

Table 7b. The Three initial Intervals in astronomical context.

However, an Old Babylonian text (CBS 8165)⁷⁴ begins with data for these same three cubes described as follows by Neugebauer and Sachs (1945:34). The missing four cubes (out of seven) from VAT 8547) are suggested on the right.

No. 32. CBS 1865. Fragment of a single table.
Obverse cube roots beginning with

1e-1 ba-si₈
8e-1 ba-si₈
27e-1 ba-si₈

Only **parts of seven lines** are preserved. The Reverse is inscribed with scattered, half-erased numbers which are obviously connected with the calculation of cubes, e.g., 2,13,20 = (20³) [emphases added].

No. 32. CBS 1865. single table, 1 through 216?
Obverse cube roots beginning at 1 with additions to 343, the septenary.

1e-1 ba-si₈ Unity
8e-1 ba-si₈ Double Interval
27e-1 ba-si₈ Triple Interval
64e-1 ba-si₈ Quadruple Interval
125e-1 ba-si₈ Quinary Interval
216e-1 ba-si₈ Senary Interval
343e-1 ba-si₈ Septenary Interval

Even so, apart from Earth, with the Double interval uncorrelated, the periods begin with that of the outermost known planet Saturn, and although the last period of 216 years includes the periods of Neptune (163.7232 years) and Uranus (83.7474 years), there is little to connect these values with the next three intervals. However, certain numerical properties for this set are known from later works, including the relation $a^3 + b^3 + c^3 = d^3$.³ Thus, for the sequence $27 + 64 + 125 = 216$, $a = 3$, $b = 4$, $c = 5$ and $d = 6$, which are the required cube roots of the four Intervals in question. Furthermore, it turns out that *the distances for Uranus and Neptune are sequentially inherent in this set* by way of the *geometric means*, initially from the products for the distances, e.g., between the *Fourth* and *Fifth* Intervals ($4 \cdot 5 = 20$) for the distance (R) of Uranus, and between the *Fifth* and *Sixth* Intervals ($5 \cdot 6 = 30$) for the distance (R) of Neptune. Both followed in turn by the square roots of the two distances for Vi to complete the geometric means with (T), the corresponding periods obtainable from Vi^3 or $Vi \cdot Vi^2 = T$, etc.,. But either way, note the 30 ("a.u") for the distance of Neptune is the candidate for the "completing the square" exercise and the ellipse for this planet; likewise the 20 (a.u.) for Uranus. and the 9 (a.u.) for Saturn.

1 Point ⁰	Vi Line ¹	R Square ²	T Cube ³	[Unity, Velocity, Distance, Period] Vi supplies the names for the Intervals.	
1	1	1	1	Earth, Single Interval, T, R, Vi, Vr	
1	✓2	2	✓2.2	(Gm,) T = 2.828427 years	
1	2	4	8	Double Interval, Binary.	
1	✓6	6	✓6.6	(Gm), T = 14.69693 years	
1	3	9	27	Saturn, Triple Interval, Ternary.	
1	✓12	12	✓12.12	(Gm), T = 41.56922 years	
1	4	16	64	Quadruple Interval, Quaternary.	
1	✓20	20	✓20.20	Uranus (Gm), T = 89.44272 years	
1	5	25	125	Quintuple interval, Quinary.	
1	✓30	30	✓30.30	Neptune (Gm), T = 164.3168 years	
1	6	36	216	Sextuple Interval, Senary.	

Table 7c. The Six integer Intervals & five Geometric means.

Whereas at another time and place the fact that the Double [1,2,4,8] and Triple [1,3,9,27] Intervals predominate in ancient writings includes the condensed arrangement by Proclus in his *Commentary* on the *Timaeus* of Plato:⁷⁵

"the terms 2, 3, 4, **9**, 8 and 27 with reference to the first part, ascribes the principal dignity to the monad."

also noteworthy for the positioning of 9 *before* 8, but understandable with the inclusion of the monad {1} as the standard frame of reference. Thus, although doubled-up, the following familiar arrangements:

1 Point ⁰	V_i Line ¹	R Square ²	T Cube ³
[{1}],	2, 3,	4, 9,	8, 27]

7d. Proclus: Combined Intervals.

with the “squaring and cubing” of $(V_i) = 2$ and $(V_i) = 3$ providing the parameters for *both* sets of (R) and (T) .

Lastly, there remains the place of the *Double* interval in all this, for apart from joining with the triple interval as the first integer sets that describe the harmonic law there is no planet *per se* that fits this first interval. Furthermore, the understanding that unity provides the frame of reference is also a necessity for this application. In this respect Macrobius (ca.400 CE) in *Commentary on the Dream of Ciprio*, states: “The *monad* represents the point because, like the point, which is not a body but which produces bodies from itself.”⁷⁶ Emphasis on “bodies” may seem strange, but *astronomical bodies* and their *distances* fits well enough, especially since Macrobius, commencing with 2, next combines all the double and triple numbers to determine “solid bodies.” This process includes the cubing of both 2 and 3, followed by the statement that: “In either case the *monad* is necessary to produce a solid body.” Thus for the Double, Triple and all such intervals the *monad* (unity) serves as the first parameter and frame of reference, and the second (V_i) provides both the *name and the number* to be *squared and cubed* for mean distances and periods as stated earlier. Finally, for expository purposes at least, with the radius (or the heliocentric distance) of Earth 1, and *diameter* of 2, for the double interval [1, 2, 4, 8] the corresponding *cube* - length of side 2, area of one surface 4 (V_i^2) and volume of 8 (V_i^3) - are all realized. As, perhaps, similarly reflected in the following assignment by Philolaus:⁷⁷

The number 8, which the arithmeticians call the first actual cube, has been given by the Pythagorean Philolaus the name of geometrical harmony, because he thinks he recognizes in it all the harmonic relations.

and the following extended summary:⁷⁸

(Cassidorus, Exp. in Ps., p. 36).

[According to Pythagoras] . . . Number is the first principle, a thing which is undefined, incomprehensible, having in itself all numbers which could reach infinity in amount. And the first principle of numbers is in substance the first Monad, which is a male monad, begetting as a father all other numbers. Secondly, the Dyad is a female number, and the same is called by the arithmeticians even. Thirdly, the Triad is a male number; this the arithmeticians have been wont to call odd. Finally, the Tetrad is a female number, and the same is called even because it is female. . . . Pythagoras said this sacred Tetractys is: ‘the spring having the roots of ever-flowing nature!’ . . . the four parts of the Decad, this perfect number, are called number, monad, power and cube.

(Hippol., Phil., 2. Dox. 355).

XVIII. The Pythagorean Tetractys and the point-line-area-volume analogy

However, it becomes apparent that the “Double” and “Triple” intervals in Plato’s *Timaeus* and the point, line, area, volume analogy are also linked to the Pythagorean *Tetractys*. Which can now be considered in terms of the “lines” and the “areas” in the present “completing the square” exercise. In fact, the first line “I have subtracted the side of my square from the area” leads directly to the groupings of four in the Pythagorean *Tetractys*, beginning simply enough with the first four integers as expanded further in the following commentary by Thomas Taylor:⁷⁹

The first is that which subsists according to the composition of numbers.
The second, according to the multiplication of numbers.
The third subsists according to magnitude.
The fourth is of the simple bodies.
The fifth is of figures.
The sixth is of things rising into existence through the vegetative life.
The seventh is of communities.
The eighth is the judicial power.
The ninth is of the parts of the animal.
The tenth is of the seasons of the year.
And the eleventh is of the ages of man.

All of them however are proportional to each other. For what the monad is in the first and second tetractys, that a point is in the third; fire in the fourth; a pyramid in the fifth; seed in the sixth; man in the seventh; intellect in the eighth; and so of the rest. Thus, for instance, the first tetractys is 1. 2. 3. 4. The second is the monad, a side, a square, and a cube. The third is a point, a line, a superficies, and a solid. The fourth is fire, air, water, earth. The fifth the pyramid, the octahedron, the icosahedron, and the cube. The sixth, seed, length, breadth and depth. The seventh, man, a house, a street, a city. The eighth, intellect, science, opinion, sense. The ninth, the rational, the irascible, and the epithymetic parts, and the body. The tenth, the spring, summer, autumn, winter. Eleventh, the infant, the lad, the man, and the old man.

The world also, which is composed from these tetractys, is perfect, being elegantly arranged in geometrical, harmonical, and arithmetical proportion; comprehending every power, all the nature of number, every magnitude, and every simple and composite body. But it is perfect, because all things are the parts of it, but it is not itself the part of any thing. Hence, the Pythagoreans are said to have first used the before-mentioned oath, and also the assertion that “all things are assimilated to number.”

Thomas Taylor, *IAMBLICUS’ LIFE OF PYTHAGORAS*

As for the formulation of the harmonic law in terms of points, lines, areas and volumes, this is at least systematic in contrast to the routine acceptance of the surprising fact that in the Solar System the cube of a radius vector is equal to the square of the time the latter takes to complete 360° about the “center.” Small wonder, then, that one finds included in ancient writings, e.g., Plato’s *Epinomis*, the following condensed commentary concerning: ⁸⁰

.. what is called by the very ludicrous name *mensuration*, but is really a manifest assimilation to one another of numbers which are naturally dissimilar, effected by reference to areas. Now to a man who can comprehend this, it will be plain that this is no mere feat of human skill, but a miracle of God’s contrivance. Next, numbers raised to the third power and thus presenting an analogy with three-dimensional things. Here again he assimilates the dissimilar by a second science, which those who hit on the discovery have named *stereometry* [the gauging of solids], a device of God’s contriving which breeds amazement in those who fix their gaze on it and consider how universal nature molds form and type by the constant revolution of potency and its converse about the double in the various progressions. The first example of this ratio of the double in the advancing number series is that of 1 to 2; double of this is the ratio of their second powers [1 : 4], and double of this again the advance to the solid and tangible, as we proceed from 1 to 8 [1, 2, 2², 2³]; the advance to a mean of the double, that mean which is equidistant from lesser and greater term [the arithmetical], or the other mean [the harmonic] which exceeds the one term and is itself exceeded by the other by the same fraction of the respective terms— these ratios of 3 : 2 and 4 : 3 will be found as means between 6 and 12—why, in the potency of the mean between these terms [6, 12], with its double sense, we have a gift from the blessed choir of the Muses to which mankind owes the boon of the play of consonance and measure, with all they contribute to rhythm and melody. So much, then, for our program as a whole. But to crown it all, we must go on to the generation of things divine, the fairest and most heavenly spectacle God has vouchsafed to the eye of man.

PLATO, *Epinomis* 990c -991b, trans. A. E Taylor (1531-1532)

All of which sheds some light on the emphasis placed on the latter, the triple interval and their relationship to the *Timaeus*, as Macrobius explains at the end of a lengthy passage after introducing the number 7 alone and also in association with 8. The passage concludes by assigning an all-encompassing significance to the number 5 while also providing points of immediate relevance with a hint of more to follow: ⁸¹

[Chapter V, Macrobius, *Commentary on the Dream of Scipio*.]

[12] Thus it becomes clear that numbers precede surfaces and lines (of which surfaces consist), and in fact come before all physical objects. From lines we progress to numbers, to something more essential, as it were, so that from the various numbers of lines we understand what geometrical figures are being represented.

[13] But we have already remarked that surfaces with their lines are the first incorporeality after the corporeality of bodies and that they are nevertheless not to be separated from bodies on account of their indissoluble union with them. Therefore, whatever precedes surface is purely incorporeal; but we have shown that number is prior to surface and to lines; hence the first perfection of incorporeality is in numbers, and this is, as we previously stated, the common perfection and fullness of all numbers.

[14] The fullness of those numbers which form bodies or bind them together is a particular one, as we suggested above and will explain shortly; at the same time I shall not deny that there are other reasons for numbers being full.

[15] That the number eight produces a solid body has been demonstrated above. But this number has a special right to be called full, for in addition to its producing solid bodies it is also without doubt intimately related to the harmony of the spheres, since the revolving spheres are eight in number. More about this later, however.

[16] All numbers which, when paired, total eight are such that fullness is produced from their union. The number eight is either the sum of the two numbers that are neither begotten nor beget, namely, one and seven, whose qualities will be discussed more fully in their proper place; or from the doubling of that number which is both begotten and begets, namely, four, since four comes from two and produces eight; or from three and five, one of which is the first uneven number of all, while the characteristics of the other will be treated later. [End of Chap. V].

Chapter Vi, Macrobius, *Commentary on the Dream of Scipio*.⁸²

[1] FURTHERMORE, a reason patent to all persuades us that the number seven also deserves to be called full. But we cannot pass over this fact without first expressing admiration that of the two numbers which, when multiplied with each other, determine the life span of the courageous Scipio, the one is even, the other odd. Indeed, that is truly perfect which is begotten from a union of these numbers. An odd number is called male and an even female; mathematicians, moreover, honor odd numbers with the name Father and even numbers with the name Mother.

[2] Hence *Timaeus*, in Plato’s dialogue by the same name, says the God who made the WorldSoul intertwined odd and even in its make-up: that is, using the numbers two and three as a basis, he alternated the odd and even numbers from two to eight and from three to twenty-seven.

[3] The first cubes in either series arise from these: using the even numbers, two times two, or four, make a surface, and two times two times two, or eight, make a solid; again, using the odd numbers, thrice three, or nine, make a surface, and three times three times three, or twenty-seven, the first cube.

Chapter Vi, Macrobius, *Commentary on the Dream of Scipio*. (continued).

Accordingly we are given to understand that *these two numbers, I mean seven and eight, which combine to make up the life-span of a consummate statesman, have alone been judged suitable for producing the World-Soul for there can be no higher perfection than the Creator.*

[4] This, too, must be kept in mind, that in affirming the dignity belonging to all numbers we showed that they were prior to surfaces, lines, and all bodies; and besides we learned a moment ago that *numbers preceded the World-Soul, being interwoven in it, according to the majestic account in the Timaeus, which understood and expounded Nature herself.*

[5] Hence the fact which wise men have not hesitated to proclaim is true, that *the soul is a number moving itself. Now let us see why the number seven deserves to be considered full on its own merits.* That its fullness may be more clearly realized, let us first examine the merits of the numbers whose sums make up seven, then, at last, the capabilities of seven itself.

[6] The number seven is made up either of *one and six, two and five, or three and four. It would be well to treat these combinations separately; we will confess that no other number has such a fruitful variety of powers.*

(Arguments [7] through [18] omitted)

....

[19] *The possession of unusual powers came to the number five because it alone embraces all things that are and seem to be. (We speak of things intelligible as "being," and of things material as "seeming to be," whether they have a divine or a mortal body.)" Consequently, this number designates at once all things in the higher and lower realms. (emphases supplied)*

Macrobius, *Commentary on the Dream of Scipio*, Chapter Vi, Trans by William Harris Stahl. (emphases added)

In point [2] of Chapter Vi, the use of "the numbers two and three as a basis" to construct the "World-Soul" in Plato's *Timaeus* is by now familiar in so much as it incorporates the single line of Proclus (Fig. 4d) :

[2] Hence *Timaeus*, in Plato's dialogue by the same name, says the God who made the World-Soul intertwined odd and even in its make-up: that is, *using the numbers two and three as a basis, he alternated the odd and even numbers from two to eight and from three to twenty-seven.*

with later statements by Macrobius which refer to seven, a number, which also "deserves to be considered full on its own merits" ([5]).

Further, the number seven is also included as the sum of one and six, two and five, plus (finally) *three and four*, followed by the injunction that "it would be well to treat these combinations separately; we will confess that no other number has such a fruitful variety of powers." ([6]). Grouped in this manner it is difficult *not to* suspect that the Lucas series, i.e., 1,3,4,7,... is part of the dialogue, except, of course, that Francois Lucas (1842-1891) is credited with the (belated) "discovery" of this elementary series.

As for the bald and definite statement in [3] that the numbers seven and eight "combine to make up the life-span of a consummate statesman (and) have alone been judged suitable for producing the World-Soul," this may, perhaps, refer to Plato's *Statesman*. The latter begins with a laborious emphasis on the number three, and later seeks "to define a perfect constitution" from "the rule of one, the rule of the few, and the rule of the many." This is followed by somewhat obscure instructions; "Dividing each of the three into two let us make six, having first separated the true constitution from all, calling it the seventh," noting next that "Under the rule of one we get kingly rule and tyranny; under the rule of the few, as we said, come the auspicious form of if, aristocracy, and also oligarchy."⁸³

It is at this point that the "Number of the Tyrant" in Plato's *Republic* may be considered, with both the latter and the present topic treated in terms of the eighth Pythagorean *tetractys* - "judicial power." Here, according to Macrobius, the number of the (consummate) statesman is explicitly defined as the product of 7 and 8 = 56 years.

This period is readily associated with the second perfect number, 28, or more precisely, twice this value (56). This is an elementary matter, to be sure, but 28 years lies well within the range of periods provided by the apsidal data for Saturn (approximately 27 to 32 years) and also, with 2 periods of revolution for Saturn in 56 years, there will be 3 synodic lap cycles (SD1) and 5 revolutions for Jupiter to complete the (2 : 3 : 5) resonant triple for these two major planets. Furthermore, continuing in the same vein, comprehension is gained from the understanding that the base period of the Pierce planetary framework (and/or its fore-runners) will be 6 times that of the period of Saturn.

Therefore the required base period *P1* is the product of the first and second perfect numbers (6•28 = 168 years) which can then be applied in standard fashion to the Pierce divisors shown below in Table 5.

Here, however, the correlation with the *Phi*-series begins to deteriorate beyond Jupiter but continues to improve below this planet *en route* to Mercury:

PLANETS N Synodics #	RATIOS (Pierce)	DIVISORS (added)	RES.TRIPLES [(RZT)]	PERIODS (Days) P1/Divisors (JYR)	PERIODS (Years) P1/Divisors (JYR)	Phi-series $T = \phi^x$ Exponents: x	
Neptune 1	1/1	1		61,362	168		
Synodic 2-1	1/1	1	1(1)2	61,362	168		
Uranus 2	1/2	2		30,681	84		
Synodic 3-2	1/2	4	1(2)3	15,340.5	42		
Saturn 3	2/3	6		10,227	28	29.03444185	7
Synodic 4-3	2/3	9	2(3)5	6,818	18.666*	17.94427191	6
Jupiter 4	3/5	15		4,090.8	11.2	11.09016994	5
Synodic 5-4	3/5	25	3(5)8	2,454.48	6.72	6.854101966	4
M-J Gap 5	5/8	40		1,534.05	4.2	4.236067977	3
Synodic 6-5	5/8	64	5(8)13	958.78125	2.625	2.618033989	2
Mars 6	8/13	104		590.01923	1.6153846154	1.618033989	1
Earth/Syn 7-6	8/13	169	8(13)21	363.08876	0.9940828402	1.000000000	0
Venus 7	13/21	273		224.76923	0.6153846154	0.618033989	-1
Synodic 8-7	13/21	441	13(21)34	139.14286	0.3809523809	0.381966011	-2
Mercury 8	21/34	714		85.911765	0.2352941176	0.236067978	-3

Table 8. Pierce planetary framework, Solar System & Phi-series. Base period from perfect numbers $6 \cdot 28 = 168$ years.

IXX. More on the Triple Interval and mathematical astronomy in Plato's Dialogues

The process of squaring and cubing to determine distances and periods is not only simpler than using fractional exponents, it also lends itself to easy dissemination, especially the integer parameters associated with the triple interval, e.g., [1, 3, 9, 27]: Unity (1), Velocity Vi (3), Distance R (9) and corresponding period at perihelion (27 years) as applied to Saturn, the outermost planet known in Antiquity. Which in this form plays a role in familiarization as exemplified by the condensed details in Plato's *Republic IX*, 587d–588a.⁸⁴ "Three times three, then, by numerical measure," that "by longitudinal mensuration (is) a plane number," followed "by squaring and cubing," after which "it is clear what the interval of this separation becomes." Also, by "taking it the other way about," and if one "tries to express the extent of the interval . . . he will find on completion of the multiplication that he lives 729 times as happily and that the tyrant's life is more painful by the same distance." Understandably, these lines represent "An overwhelming and baffling calculation . . . and what is more, it is a true number and pertinent to the lives of men if days and nights and months and years pertain to them." Thus not only the same process, but also, it would seem, the triple interval with both time and distance incorporated in the dialogue, albeit obliquely. Here, commencing with Vi² the distance 9 (a.u.) is followed by "squaring and cubing" and the common product 729 ($T^2 = 27^2$ and $R^3 = 9^3$) included for good measure. On the other hand, the confusion caused by references to: "life," "the tyrant" and "pain and pleasure" in the above become somewhat more understandable when the methodology inherent in the Pythagorean *Tetractys* is taken into consideration.

But above all else, it is the suitability of the Triple interval [1, 3, 9, 27] - Ref. Unity, velocity Vi = 3, Distance 9 (a.u.) and corresponding period of revolution of 27 years applied to Saturn which is the most helpful. Then again, the data pertain to the shortest period (27 years) associated with perihelion, and not the mean period or longest period at aphelion, even though the latter was the result of the "completing the square" approach to elliptical orbits. Nor for that matter does rounding the aphelion period to 32 years help greatly either since the distance 10 (a.u.) now becomes 10.079368 (a.u.). And also, multiplication of the velocity (3.162277) by 10, the corresponding distance to produce the period T of 31.622777 years is also lost. Although not lost entirely, since in condensed form it is again relation (4) from the 1989 Galileo paper discussed earlier, i.e., $T = Vi^3$.

APSIDES	Unity	Velocity Vi	Distance R	Period T
Perihelion	1	3	9	27
SATURN	1	3.0822070	9.5	29.280967
Aphelion	1	3.1622777	10	31.622777
Perihelion	1	3	9	27
SATURN 2	1	3.0898733	9.547317	29.5
Aphelion	1	3.1748021	10.079368	32

Table 9. The Triple Interval and integer values

Whereas the 32-year aphelion period assigned to Saturn reappears among the workings of the “Alchemists” in the far from straightforward context supplied below. Although still a difficult text, towards the end some degree familiarity with material presented here in the preceding pages becomes apparent. In particular, the question:

“If the Capacity or Circumference of the sphere be 32 foot, how much will one of the sides of the Cube be to Equalize the Capacity of this Sphere?”

at least begins to swing the dialogue towards the “Point-Line-Area-Volume” analogy, “completing the cube,” and a reference to Saturn, plus “six the first of the perfect numbers.”

Atalanta fugiens emblems 21 - 25

Michael Maier's alchemical emblem book *Atalanta fugiens* was first published in Latin in 1617. It was a most amazing book as it incorporated 50 emblems with epigrams and a discourse, but extended the concept of an emblem book by incorporating 50 pieces of music the 'fugues' or canons. In this sense it was an early example of multimedia.⁸⁵

Emblem XXI. [transcribed by Hereward Tilton]

Make of the man and woman a Circle, of that a Quadrangle, of this a Triangle, of the same a Circle and you will have the Stone of the Philosophers.

.....

[Closing paragraphs]

But that this was not unknown to the Philosophers of Nature is apparent from this: That they command a Circle to be turned into a Quadrangle, and this by a Triangle to be reduced again to a Circle. By a circle they understand the most simple body without angles, as by the Quadrangle they do the four Elements. It is as if they should say: The most simple corporeal Figure that can be found is to be taken and divided into four Elementall Colours, becoming an Equilaterall Quadrangle. Now every man understands that this Quadrangle is Physicall and agreeable to Nature, by which far more benefit accrues to the Publick, and more light appears to the mind of Man, than by any meere Theory of Mathematicks when abstracted from Matter. To learn this perfectly a Geometrician acting upon solid bodies must enquire what is the depth of solid Figures, as for example the Profundity of Sphere and Cube must be knowne and transferred to manuell use and practice. If the Capacity or Circumference of the sphere be 32 foot, how much will one of the sides of the Cube be to Equalize the Capacity of this Sphere? On the contrary, one might look back from the Measures which the Cube contains to the feet of each Circumference.

In like manner the Philosophers would have the Quadrangle reduced into a Triangle, that is, into a Body, Spirit and Soul, which three appear in the three previous colours before Rednesse: that is, the Body or earth in the Blacknesse of Saturn, the Spirit in the Lunar whitenesse as water, and the Soul or air in the Solar Citrinity. Then the Triangle will be perfect, but this again must be changed into a Circle; that is, into an invariable rednesse, by which operation the woman is converted into the man and made one with him, and six the first of the perfect numbers is absolved by one, two having returned again to an unity in which there is Rest and eternall peace.

Closing paragraphs for *Atalanta fugiens* emblem 21 by Michael Maier (1617). Introduced by Adam Mclean.⁸⁵

The cryptic description of Emblem XXI “*Make of the man and woman a Circle, of that a Quadrangle, of this a Triangle, of the same a Circle and you will have the Stone of the Philosophers*” is discussed in Part Four with respect to isosceles and equi-lateral triangles emphasized in Plato's *Timaeus*. Linked to the “Rotation of the Elements,” it is shown that the former pertain to the Fibonacci series with ratios between the sides of the triangles increasing towards *Phi* itself as the rotations continue. The Lucas series variant makes use of the same rotations but differs in range and purpose as it continues to approximate the $\frac{1}{2}\text{Phi}$ -series, improving towards this outcome with each successive rotation.

XX. The Body, Spirit and Soul Triad

But there is also something else of importance, which is the reference to “a triangle” reduced into “a Body, Spirit and Soul.” Of the three, “body” is perhaps the simplest since it is a well attested term for an astronomical body. Furthermore, with this key providing the way, there are sufficient examples in the literature to also assign “spirit” to Velocities (*V_i* & *V_r*) and “soul” to Time (*T_s*), but not without some difficulty in the absence of additional information.

In terms of the extant alchemical literature, Philip à Gabella (1615) perhaps leads the way by passing on that:⁸⁶

according to the first fathers of philosophy the magical contemplation of **the ternary** encompassed **body, spirit and soul**

while in more detail, Johann Isaac Hollandus in *Of natural & supernatural things*. London, 1670, writes:⁸⁷

They say, in our Stone are the four Elements, and they say true; for the four Elements must be separated out of **Saturn**. They say, our Stone consists of **Soul, Spirit and Body**, and these three become one. They say true; when it is made fixed for the white **Mercury** and Sulphur with its' Earth, then these three are one.

and in *Paracelsus his Aurora, & Treasure of the Philosophers* (1659) where “Mercurius” states in part:⁸⁸

“This mystery it is permitted only to the prophets of God to know. Hence it comes to pass that this Stone is called animal, because in its blood a **soul** lies hid. It is likewise composed of **body, spirit, and soul**. For the same reason they called it their microcosm, because it has the likeness of all things in the world, and thence they termed it animal, as Plato named the great world an animal.”

plus more from *The naturall Chymicall Symbol or short Confession of Doctor Kunwrath*:⁸⁹

[Three in One, One in Three]. But this is the true philosophical doctrine of the philosophers **Mercury**. That Three is One general Chaos, Three in essence, namely **Body**, **Soule**, and **Spirit**; and these Three Essences are had in One substance or thing and neere at hand.

[**Body Soule Spirit**.] There is one Essence of the **Body** one other of ye **Spirit**, one other of the **Soule**; But ye **Body**, **Soule**, and **Spirit** are one thing, wherein all the three are together equally necessarily present at the same time.

For the **Body** is not made by the Arte of Man, nor is the **Spirit** made by the Arte of Man, neither is the **Soule** made by the Arte of Man.
(all emphases supplied)

Macrobius, on the other hand, in his *Commentary on the Dream of Scipio* is more direct while at the same time invoking the Double and Triple intervals with emphasis on assigning the last exponent of the Point-Line-square cube analogy to a solid body, then finally arriving at Soul = time and motion "with its animating power" and non-corporeal form: ⁹⁰

[12] Since the monad is the source of even and uneven numbers alike, the number three should be considered the first line. This tripled gives nine, which from its two lines, as it were, produces a **body** with length and breadth, as was the case with the number four, the second of the even numbers. In the same way the number nine tripled supplies the third dimension. Thus with the uneven numbers a solid **body** is formed in twenty-seven, three times three times three, just as with even numbers eight or two times two times two made a **solid body**.

[13] In either case the **monad** is necessary to produce a **solid body**, in addition to the other six numbers, three even and three uneven, two, four, and eight being the even numbers and three, nine, and twenty-seven the uneven.

[14] Plato's *Timaeus*, in disclosing the divine plan in the creation of the World-Soul, said that the **Soul** was interwoven with those numbers, odd or even, **which produce the cube or solid**, not meaning by this that the **Soul** was at all corporeal; rather, in order to be able to penetrate the whole world with its animating power and fill the **solid body** of the universe, the **Soul** was constructed from the numbers denoting solidity.
(all emphases supplied)

A small selection, to be sure and also not particularly compelling, but the complexity of the matter can at least be given a detailed basis in the form of the *Phi*-series planetary framework generated earlier in Table 3. As shown below, a reduced, rearranged version of the latter has double or triple occurrences of parameters in each of the three categories - Soul (Periods *T*), Body (Distances *R*) and Spirit (Velocity *Vi*) with (*Vr*) not part of the set but available:

PLANETS N	Unity	SPIRIT (<i>Vi</i>)	BODY (<i>R</i>)	SOUL (<i>T</i>) = ϕ^x	Velocity (<i>Vr</i>)
Synodics #	Point, <i>Vi</i> ⁰	Line, <i>Vi</i> ¹	Square, <i>Vi</i> ²	Cube, <i>Vi</i> ³	(Reciprocal <i>Vi</i>)
Saturn 3	1	3.073532624	9.446602789	29.03444185	7 0.325358512
Synodic 4-3	1	2.618033989	6.854101966	17.94427191	6 0.381966011
Jupiter 4	1	2.230040414	4.973080251	11.09016994	5 0.448422366
Synodic 5-4	1	1.899547627	3.608281187	6.854101966	4 0.526441130
M-J Gap 5	1	1.618033989	2.618033989	4.236067977	3 0.618033989
Synodic 6-5	1	1.378240772	1.899547627	2.618033989	2 0.725562630
Mars 6	1	1.173984997	1.378240772	1.618033989	1 0.851799642
Earth/Syn 7-6	1	1.000000000	1.000000000	1.000000000	0 1.000000000
Venus 7	1	0.851799642	0.725562630	0.618033989	-1 1.173984997
Synodic 8-7	1	0.725562630	0.526441130	0.381966011	-2 1.378240772
Mercury 8	1	0.618033989	0.381966011	0.236067978	-3 1.618033989

Table 3s. *Phi*-series *Vr*, *Vi*, *R*, *T* and the Point-Line-Square-Cube / Spirit-Body-Soul assignments.

XXI. The World-Machine of Francesco Giorgi (1525).

The significance of certain sets of data is more apparent for some than others; for example, the primary numbers for the Synodic 6-5 position are, from right to left the three constants assigned to the mean planetary Periods, *i.e.*, relation (8), the Distances of the Planets, relation (9) and the Periods for the planet-synodic expansions, relation (12). Other assignments of interest include the three locations for *Phi* itself, firstly as the Period (SOUL) of Mars, secondly as the Velocity *Vr* of Mercury and thirdly also the Velocity (*Vi*) (SPIRIT) for the Mars-Jupiter Gap. Whereas the three occurrences for the synodic 8-7 period *T* (SOUL) between Venus and Mercury are the Distance *R* (BODY) for Mercury, and again the Velocity (*Vr*) for synodic4-3 between Jupiter and Saturn. Which, in addition to dynamic aspects, also involves the first of unities ("384") in a summation concerning the construction of a "world-machine" by Francesco Giorgi (1466-1540) whose *Harmonia Mundi* (1525) concludes with a remarkably condensed statement apparently concerned with the construction of a "world-machine" as opposed to the "world-soul" of Plato: ⁹¹

Thus this world-machine, consisting of a simple number squared and cubed, leads to a sovereign concord, not only of the ternary to 27 but of the senary to 162 and from 384 to 10,368, as we have set them out above.

Francesco Giorgi, *Harmonia Mundi* (Venice, 1525)

The statement is, however, readily decoded, especially if the reader is familiar with both the “Ternary” and “Senary” (the Triple interval [1, 3, 9, 27] and the Sextuple interval [1, 6, 36, 216] respectively) in the present context. This said, it appears that there are two significant activities involved, firstly with the *Phi*-series concerning the number “27” and the range “from 384 to 10,368” which concerns heliocentric planetary distances. And secondly, a separate approach which involves providing the upper limit for the outermost planet (“Neptune”) of 162 (years) as given.

Remaining with the latter there is therefore now a further base period for the Pierce planetary framework and its divisors. Although the resulting period for Mercury is low ($162/714 = 0.227209$ years) the division by 6 for Saturn, the outermost *known* planet in Antiquity ($162/6$) results in 27 years, which is also the period/volume parameter (or Soul again in years) of the Ternary. Whereas the product: $27 \cdot 384 = 10,368$, the range assigned in the text. Except for one thing, which is to provide a meaning for this data the range is best understood to extend from 0.384 “a.u.” to 10.368 “a.u.” in 27 steps. As for the hows and the whys of this matter, both are influenced by an earlier 36-step sequence, which is fortunately explained in the meticulous footnotes accompanying another relatively obscure work, this time by George Burges entitled: *THE TREATISE OF TIMEUS THE LOCRIAN ON THE SOUL OF THE WORLD AND NATURE* (London, 1876).

Here Burges explains the relationship (and more) between the 36-step and 27-step configurations augmented by recognizable references to the Binary [1,2,4,8], Ternary [1,3,9,27] and the point-line-square-cube analogy. Plus something else, which is an extension to incorporate the “Soul of the World.”⁹²

... But why were the terms fixed at 36 ? The reason is to be found in the mysteries of the school of Pythagoras, where it was thought proper to multiply 384, the assumed term. By 27. But why 27? Because that number is the sum of the first numbers, which must represent lines, surfaces, solids, squares, and cubes, added to unity. Thus 1 is unity ; 2 and 3, the first numbers representing lines : 4 and 9, the first surfaces, and both squares, the former of an even number (2), and the latter of an odd number (3). Taking then the number 27 as the symbol of the world, and the numbers which it contains as the symbols of the elements and their combinations, it was reasonable for the Soul of the world, which is the very basis of order and of combinations, which constitute the world, to be composed of the same elements (of order) as the number 27 in itself.

George Burges, Supplement to *TIMEUS THE LOCRIAN* (1876 : 172-173)

This leaves the matter of the “First of Unities” - (“384”) also discussed by Burges in *TIMEUS THE LOCRIAN, The Works of Plato : A New Literal Version* Vol. VI. (1876: 150-151), which in a technical, dynamic sense turns out to be far from simple. It is also, as a variation of other interpretations of Plato’s *Timaieus*, impressive in the manner by which the “Body” = Distance (Square), and “Soul” = Period (Cube) methodology discussed earlier is extended and linked to the “First of Unities.”⁹³

But the soul of the world has (the deity) united with the centre and led it outwards, investing the world wholly with it, and making it a mixture of Form undivided, and of Substance divided, so as to become one mixture from these two; for which (world) he mixed up two threes, the origin of motion, one connected with the same, the other with the different; which (soul), being mixed with difficulty, was mixed not in the easiest way. Now all these proportions are combined harmonically according to numbers: which proportions he has divided according to a scale scientifically, so that a person is not ignorant of what things and by what means the soul is combined; which the deity has not ranked after the substance of the body, – for, as we say, that which is before is in greater honor as regards both power and time, – but he made it older by taking the first of unities, which is 384.

George Burges, *TIMEUS THE LOCRIAN* (1876 : 150-151)

To understand the First of Unities it is sufficient to know that for the *Phi*-series the “Body,” or heliocentric distance of Mercury is equal to the “SOUL,” or period of the Mercury lap-cycle with respect to Venus, i.e., Synodic 8-7. Moreover,

PLANETS N	ϕ^x	SOUL (T)	BODY (R)	SPIRIT (Vi)	Reference	
Synodics	Series	Period/Time	Distance (a.u.)	Velocity	Unity	
NAMES	#	x	Cube, Vi ³	Square, Vi ²	Line, Vi ¹	Unity Vi ⁰
Venus	7	-1	0.618033989	0.725562630	0.851799642	1
<i>Synodic</i> 8-7	-2	0.381966011	0.526441130	0.725562630		1
<i>Mercury</i> 8	-3	0.236067978	0.381966011	0.618033989		1

Table 3tr. Synodic position 8-7. *Phi*-series T, S and Vi, Mercury through Venus.

in the same wise, this number is therefore 0.384 in both applications, i.e., 0.384 a.u. for the distance and 0.384 years for the period. Which then results in the range from 0.384 to 10.368, with the latter the aphelion distance of Saturn as opposed to the lesser perihelion and mean distances. But even so, the increment for the 27 steps obtained from $(10.368 - 0.384)/27 = 9.984/27 = 0.369777^*$ with the sequence ending at 33.384316 years for the distance (BODY, R) of 10.368 a.u. at the 27th step.

The interplay between “Body” and “Soul” by Burges in this passage plus the reference to “power and time” and the odd statement that the Creator “made it older by taking the first of unities, which is 384,” is one method of placing the second instance of the parameter which occurs as the Distance of Mercury in the location of a second category. Thus also now associated with Time (Soul) and the “older” Mercury-Venus lap-cycle, Synodic 8-7 while utilising the the pheidian exponents -1, -2, -3 for the period (T, S) for Venus (T), Mercury-Venus Synodic 8-7 (S) and Mercury (T).

The reason for this odd duality - at least from a bare numerical viewpoint - is straightforward enough, simply the consequence of the *Phi*-series applied sequentially downwards to produce the periods (T, S) and then using Kepler's Third Law of planetary motion to determine the Distance of Mercury. Thus, for the *Phi*-Series, Φ^x , $x = -1 -2, -3, \dots n$) first the period of revolution of Venus = Φ^{-1} , next the period for Synodic 8-7 = Φ^{-2} and last, the period of Mercury = Φ^{-3} . However, in exponential terms, applying the conversion from Period to Distance, *i.e.*, $R = T^{2/3}$ the latter applied to Mercury Φ^{-3} results in $(\Phi^{-3})^{2/3} = \Phi^{-2}$, which is identical to the exponent which generates Synodic 8-7 and the duality.

Remarks

Just when and how this situation came to be uncovered is difficult to say, of course. Perhaps it was Pythagorean, perhaps earlier, or perhaps a little later. But in any event, there are increasingly more questions than answers at the present time, and it is likely that there are more aspects buried in the *Tetractys* which may have roles to play. For example, further divisions arising from Odd and Even numbers and their Male and Female assignments, the social, legal and political aspects, and the complexities embraced by factoring in “Life” and “Nature” itself.

Certainly among the alchemists various bits and pieces are discernable; witness now a partial agreement with the last remark above by Marcilio Ficino (1518), who states: ⁹⁴

For although sulphur and Mercury were as it were the root of metals before the first coagulation, yet now they are not, since they are brought to another nature: whence it remains that there cannot be made out of them any metallic body. Since also the chain is unknown, by which Venus & Mercury copulated together in due proportion.

Marsilius Ficinus, 'Liber de Arte Chémica', in *Theatrum Chémicum*, Vol 2, 1702.

thus a “hermaphroditic” component associated with the adjacent positions for Synodic 8-7 and Mercury a further complication. Here, however, in an extract from Bernard of Trevisan, *Le Texte d'Alchymie et le Songe-Verd*, (1695) a slightly more understandable account is offered: ⁹⁵

Therefore, my child, you see very well that I have declared all to you when I have made you understand in what manner our Sulfur is contained in the belly of the Mercury, and that it is correct to call it internal Sulfur or hidden Spirit, which is no other thing than heat and dryness, acting on the cold and the moisture, acting on the patient, the pure mercurial substance of which Sulfur is the Soul, since it is it which vivifies and sustains the Mercury which would be, without our Sulfur, only a dead, unfruitful, and sterile earth. There is then good reason to say that Sulfur and Mercury are the proper and true substances of the metals, because it is very certain that this Sulfur cannot be without Mercury and that our Mercury cannot be without this Sulfur which is intimately united and incorporated with it, as the soul is with the body.

These two names of Mercury and of Sulfur are only names for one single substance which we know under the names of Quicksilver or Mercury.

Therefore make manifest that which is hidden and make occult that which is manifest. I tell you, in that alone consists the work of the sages. Our gum curdles our milk, and our milk dissolves our gum, and they grow in the Stone of Paradise, which Stone is of two contrary natures, that is to say, of the natures of Fire and of Water.

All that I have written above ought to have opened your understanding to the intelligence of the philosophers - for what I have explained to you altogether well and have given you to understand what our Sulphur is, that the philosophers have also called Gum, Oil, Sun, Fixity, Red Stone, Curd, Safran, Poppy, Red Brass, Tincture, Dry, Fire, Spirit, Agent, Soul, Blood, Burned Brass, Red Man, and Quick Earth. I have also given you a clear and concise explanation of that which the philosophers name Water, Milk, White Wrapper, White Manna, White Urine, Cold, Moisture which does not dampen, Body, Womb, Moon, White Woman, Changing Habit, volatile, patient, Virginal Milk, Lead, Glass, White Flower, Flower of Salt, Fleece, Veil, Venom, Alum, Vitriol, Air, Wind, Rainbow, Naked Woman, and so many other names which are only for the purpose of making us conceive the qualities, properties, and the two natures of male and female contained in our substance, which is nothing else but animated Quicksilver. It is this viscous moisture mixed with its earthy part, our Mercury, and the true foundation of all our science.

It is in this great number of terms that the wise men have taken pleasure in writing their sentiment relative to our science. All these names ought to convince you of the truth of our science, for all of them have only one meaning and all of them have for their purpose only to expose the hermaphroditic Mercury to us. It is feminine if it is considered as separated from the Sulfur which it contains within it and of which it is the substance; but it is masculine if it is considered according to its Sulfur with which it is united so intimately that it cannot be separated from it; and it can be said of their marriage that they are both of them in the same flesh.

Bernard of Trevisan, *Le Texte d'Alchymie et le Songe-Verd* (1695).

But there is something else which intrudes, for in addition to the mention of Mercury and sulphur (the latter called here the soul) is what can be understood to be the third occurrence of 0.384 as an “internal sulphur or hidden spirit,” the latter part of which is undoubtedly correct, since:

$$\text{Distance (Body) of Mercury} = \text{Mercury-Venus Synodic (Soul)} = \text{Velocity (Spirit) of Jupiter-Saturn Synodic 4-3.}$$

Moreover, in all three instances and in three separate assignments the parameter for the distance, the period and the velocity is - for the *Phi*-series - the number 0.381966011. Which, if rounded at the third decimal place, becomes 0.382 and by multiplying by 1,000 the convenient integer 382. But wither and whence came the “First of Unities”? And was it something more, or possibly an alternate event after noting Marcilio Ficino’s observation that “the chain is unknown, by which Venus and Mercury copulated together in due proportion”? More intriguing is the apparent linkage between the two massive superior planets Jupiter and Saturn with the two innermost and the smallest, with Earth, perhaps errantly, somewhere in between.

On the other hand, after the positive results obtained from the mean value Pheidian and Fibonacci ratios inherent in the motions of these two gas giants, and knowing that for the *Phi*-series that the inverse velocity for Synodic 4-3 between the two is ϕ^2 with its reciprocal the relative velocity V_r , one might start by using the Fibonacci ratio $13/5 = 2.6$ versus 2.61803398 ... and take the reciprocal, or simpler yet, use the Fibonacci ratio $5/13 = 0.38461538$... Or again use 2.604 to similarly obtain 0.384024 ... which readily rounds to 0.384. It is better, however, to concentrate on the relative motions of Jupiter with respect to Saturn which already have some peculiarities in the case of the data for the former. These include an unexpected difference between the ranges for Babylonian Systems A and B for Jupiter with nothing comparable known for Saturn. How might this be linked to the First of Unities? Simply by investigating the relative motions and the two sets of data for Jupiter and Saturn to this end, but only after checks using modern data to provide a standard for their historical equivalents.

Thus for the modern Solar System, using 29.42351935 years (T_1) and 11.85652502 years (T_2) for the mean periods of revolution and applying *Phi*-series/synodic relation (1), the synodic period or lap cycle of Jupiter with respect to Saturn (S_2) is 19.85887209 years with a velocity (V_r) of 0.36927... versus the target value of 0.384. Similarly, for the Babylonian mean periods (11.86111* and 29.444* years) the synodic period (S_2) is 19.86220818 and the velocity is 0.36925, thus largely unhelpful results. But this is far from the end of the matter, especially in terms of the *Phi*-series and Babylonian methodology.

First of all, the data for the periods of revolution and the intervening synodic cycles generated by the *Phi*-series - (ϕ^x , $x = -3, -2, -1, 0, 1, 2, \dots, 7$) involve the use of *odd* and *even* exponents for the planets and synodic cycles respectively with exponents -3 and -2 for the periods of Mercury and the synodic cycle between the latter and Venus, whereas exponents 5, 6 and 7 are used for Jupiter, Synodic 4-3 and Saturn. For the Synodic 4-3 data the period is therefore ϕ^6 with the relative velocity V_r obtained from $(\phi^6)^{-1/3} = \phi^{-2} = 0.381966011$ and the Distance R for Mercury derived from $(\phi^{-3})^{2/3} = \phi^{-2}$. Lastly, the period of Synodic 8-7 remains ϕ^{-2} as generated by the *Phi*-series and incorporated in Tables 3 and 3s. Plus, of course, a more accurate though lower value for the First of Unities of 382.

Secondly, as already seen, the calculation of the mean synodic arc in Babylonian astronomy involves a further period other than the mean periods of revolution and synodic difference periods, namely the parameter P which involves both the period of revolution $P = (T - 1)$ and $P = 360/(u)$, the number of mean synodic arcs per revolution of 360° . Therefore, for Jupiter P_j is 10.86111* years and for Saturn P_s is 28.444* years with the synodic period now 17.56994909 and the corresponding velocity 0.38465... Furthermore, the same procedure applied to the modern periods likewise results in a velocity of 0.38468 ... thus purists would likely round to 0.385, but from a practical viewpoint 0.384 might nevertheless be preferred in both instances.

This is still not quite the end of this “short” excursus, because the two new synodic periods are both close to the *Phi*-series’ ϕ^6 (17.94427191) versus 17.56994909 and 17.56593320, with 1.612359016 and 1.6122975888 from the sixth roots of the new periods versus *Phi* itself. These results and others like them helped to provide the impetus for the real-time investigations of planetary motion for these two planets in particular discussed further in Part III.

So much, then for the First of Unities and the minor incursions into the complexities of past and present Alchemy leaving only one thing remaining. Which is the issue arising from the data from Aaboe’s 1964 limited research into a possible Babylonian System B augmented here by Friberg’s approach to parameters in the ratios of 80/81 and 81/80, thus reciprocals and in the format adopted for the Babylonian velocities, $M = 1;00,45^\circ$ and $u = 0;59,15,33,20^\circ$. To which can be added (in terms of symmetry) the minimum arc $m = 0;57,46,6,40^\circ$.

Next after the conversion from arcs to time and again to distance relative to unity, the ratio of the distances $(M_d - m_d)/(M_d + m_d)$, i.e., Green’s method⁹⁶ for the calculation of eccentricities, yields 0.01677279 versus 0.016708617 from modern estimates.⁹⁷

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