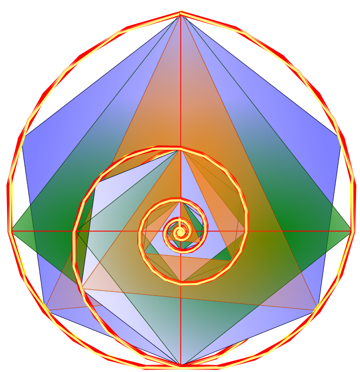
INTIMATIONS
OF
COMMONALITY
IN
PLANETARY
SYSTEMS



2023

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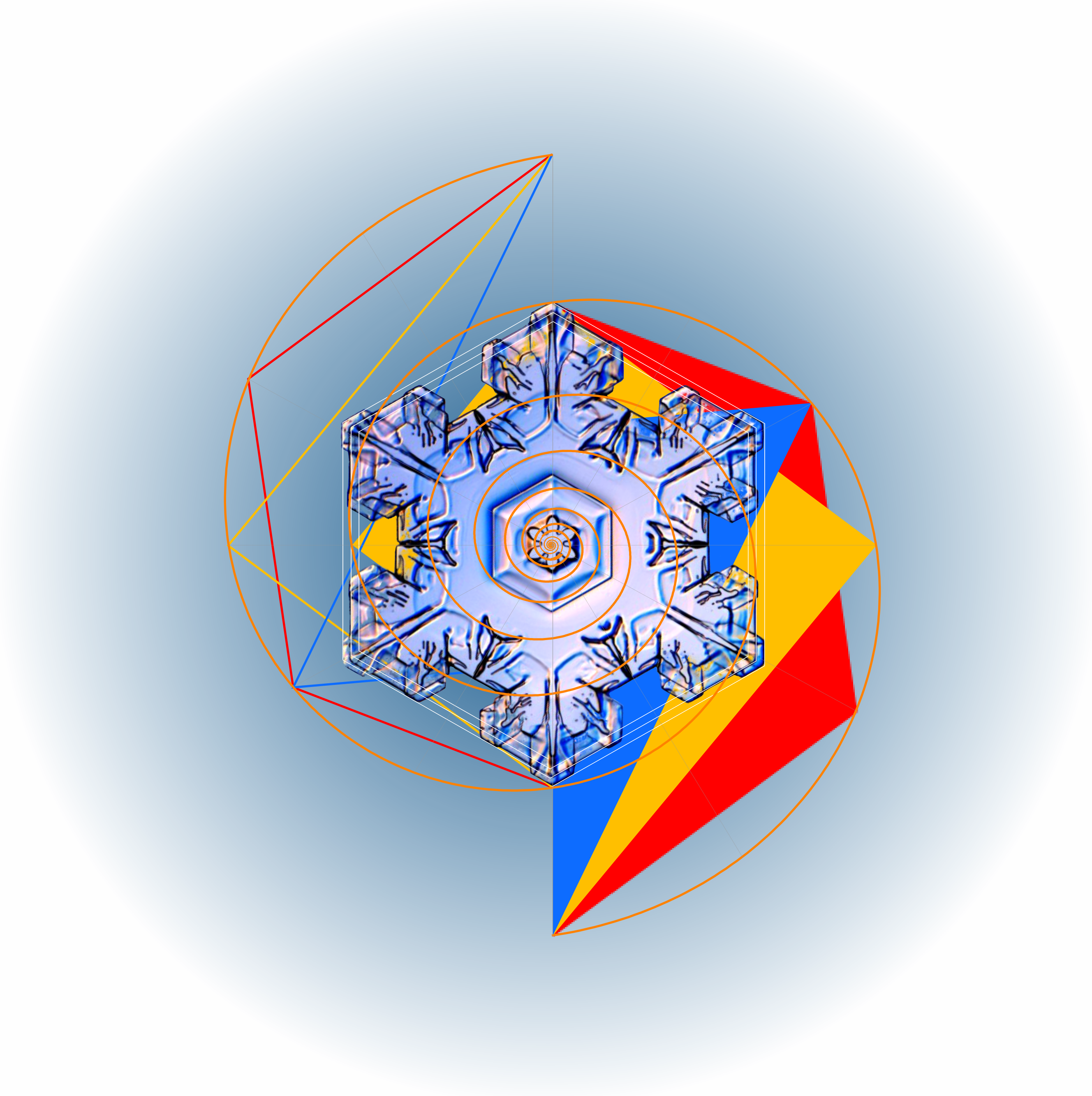
1611[1] – *"A New Year's Gift of Hexagonal Snow"* – 2023[2]



[1] *"Strena Seu de Nive Sexangula."* (Johannes Kepler's 1611 treatise *The Six-Cornered Snowflake*)

[2] Spirasolaris.ca/2023-2024

Snow Crystal: http://www.its.caltech.edu/atomic/snowcrystals/photos2/w040123b025.jpg (*sans* background)

INTIMATIONS OF COMMONALITY IN PLANETARY SYSTEMS

PART I.

The inclusion of mean synodic periods between adjacent planets completes a stalled 19th Century model of the Solar System developed by American Benjamin Pierce (1809-1880). The model suggests that the Solar System may have been subject to disruptions in the past which include the Mars-Jupiter gap, and also that *at present Earth* occupies an *intermediate*, synodic location between Venus and Mars, albeit not precisely. (Figure 1). The resulting framework also suggests that complex elements of the *Fibonacci series, Lucas series* and *Phi*-series underlie the dynamic structure of the Solar System. As a result, with twice the number of planetary periods now available, the incorporation of these three series and synodic cycles permits development of a working planetary framework for general testing. A search for further enlightenment leads to Babylonian astronomy and a wider ranging series of of inquiries with positive and negative results sufficient enough to merit the inclusion an excursus at the conclusion of Part V.

PART II.

The revised framework was applied to a variety of external planetary systems with similarities encountered (*e.g*., in the structure of HR 8799) explained in part by synodic relations common to the Solar System inherent in the *Phi*-series. Other similarities between HR 8799 and the Solar System include the possible demise of the fifth planet and possible outward shifts by both HR 8799d and HR 8799e to intermediate, synodic locations. A theoretical inward extension for this system results in a period of 0.2405942 years (HR 8799_9) versus 0.24084445 years for the Solar System's Mercury. (Figs. 1 & 2).

PART III.

Further tests lead to real-time planetary motion in the Solar System with results which confirm *pheidian* similarities already encountered with the mean periods in Part I including historical aspects explored at length in the optional *excursus*. Further concerns regarding possible 795 -year cycles for the four major superior planets gives rise to the suspicion that such matters may have significant implications while *phyllotaxis* in such contexts is also considered. (Figs.1- 6e).

PART IV.

Again searching for enlightenment Part IV deals with the two triangles in Plato's *Timaeus*, the *isosceles* and the *equilateral* triangles. It is shown the former pertains to the *Fibonacci series*, the latter to the *Lucas series* and both with respect to the "Rotation of the Elements."
The results are again *pheidian* and again apply to planetary frameworks. (Tables 1 & 2).

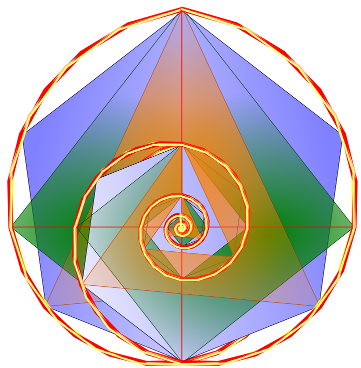
PART V.

The spiral form in time and place remains *pheidian* in both form and interest while initially concentrating on ammonites, where David Raup's neglected researches are re-examined and re-applied to ammonites after developing a series of double-precision *pheidian* test spirals. Later the study widens to include radiolarians and diodoms, but ends with concerns which are not so much about the Solar System and exoplanetary systems *per se* - but ourselves, our past, our present and our increasingly uncertain future.

With this in mind, a gentle, non-destructive option is suggested for those who might wish to embrace it.

John Nicholas Harris, West Vancouver, Canada.
29 December, 2023

PART ONE



THE PIERCE PLANETARY FRAMEWORK (1850) REVISITED

INTRODUCTION

In the 1850s American scientist Benjamin Peirce (1809–1880) produced a robust heliocentric planetary framework by applying Fibonacci-based reduction ratios to the mean periods of revolution of the eight Solar System planets.[1] Partially incomplete in dynamic terms and subjected to alternative viewpoints, this promising approach was oddly dismissed despite the attendant ramifications and total absence of any comparable planetary theory. Fortunately, however, a condensed version was at least preserved by Louis Agassiz in the latter's *Essay on Classification* (1859).[2]

As described in the latter work, Peirce began by assigning the outermost planet Neptune a convenient (albeit high) mean period of revolution of 62,000 days. Next, moving inwards, planetary periods rounded to the nearest day were derived from planet-to-planet reduction factors formed from Fibonacci numbers (1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89,144, 233, etc.), specifically, successive *alternate Fibonacci ratios* of 1/2, 1/3, 2/5, 3/8, 5/13, 8/21, 13/34 and 21/55. Thus the 62,000-day period of Neptune was reduced by one-half to obtain a mean period of revolution for Uranus of 31,000 days followed by a one-third reduction of the latter to produce a 10,333-day period for Saturn, 4,133 days for Jupiter (2/5), and so on down to an 87-day period for the innermost planet Mercury from a final reduction ratio of 13/34. However, despite this encouraging end-to-end correspondence a reduction factor for Earth was entirely absent from the alternate Fibonacci sequences. In fact, the inclusion of the latter required two *additional* reduction ratios of 8/13 and 13/21. The last ratio in Pierce's original list (21/55) remained unused but was most likely included for continuity and support for the latter's contention that "There can be no planet planet exterior to Neptune, but there may be one interior to Mercury."[3]

The Fibonacci-based reduction ratios, resulting periods and comparison with 19th Century Solar System periods were published in the *Essay on Classification* in two sparse, unlabelled tables[4] based on subdivisions of the 62,000-day period for Neptune. The initial results are shown in Table 1a with title and column assignments added:

PLANETS (ca. 1850)	PERIODS actual/days	PERIODS (reductions)	RATIOS (Pierce)
Neptune,	60,129	62,000	
Uranus,	30,687	31,000	1/2
Saturn,	10,759	10,333	1/3
Jupiter,	4,333	4,133	2/5
Asteriods,	1,200 to 2,000	1,550	3/8
Mars,	687	596	5/13
Earth,	365	366	8/13 } 8/21
Venus,	225	227	13/21 }
Mercury,	88	87	13/34

Table1a. The initial planetary structure, Peirce (1852:129)

Next, the planetary framework was extended to include *twinned ratios* provided by *adjacent* Fibonacci numbers. This produced the same periods of revolution for the planets plus intermediate periods on either side with Earth in an intermediate location between Mars and Venus. Pierce included the intermediate positions for comparable 19th Century data in the fourth column, but apart from 365 days for Earth no other intermediate periods were given. The final ratios and reductions are shown in Table 1b, again with the title and column assignments added:

PLANETS ——?——	RATIOS (Pierce)	PERIODS I (reductions)	PERIODS II (actual/days)
Neptune	1/1	62,000	60,129
"	1/1	62,000	———
Uranus	1/2	31,000	30,687
"	1/2	15,500	———
Saturn	2/3	10,333	10,759
"	2/3	6,889	———
Jupiter	3/5	4,133	4,333
"	3/5	2,480	———
Asteriods,	5/8	1,550	1,200
"	5/8	968	———
Mars	8/13	596	687
Earth	8/13	366	365
Venus	13/21	227	225
"	13/21	140	———
Mercury	21/34	87	88

Table1b. The Final planetary structure, Peirce (1852:129)

The final framework languished in this unfinished form despite correlations which included the Mars-Jupiter gap plus the possibility that planet Earth may, perhaps, be occupying an *intermediate* location. This troubling indicator should surely have been investigated, beginning, one might suggest, with mean synodic motion in general and mean synodic lap-cycles in particular.

Mean synodic motion and the intermediate periods

In fact, *all of the intermediate intervals* introduced by Pierce *are the mean synodic periods between adjacent planets*. In other words, lap-cycle times faster-moving inner planets require to complete 360° of direct orbital motion with respect to that of slower-moving outer planets. Adjacent or otherwise, mean synodic periods (*S*) between planets with mean periods of revolution T_1 and T_3 are derived from the lesser used *general synodic formula*:

$$\textit{Synodic period } S_2 = \frac{T_1 \bullet T_3}{T_1 - T_3} \quad (T_1 > S_2 > T_3) \tag{1}$$

although in modern practice relation (1) is rarely applied in this form. Synodic periods in planetary tables normally pertain to either the lap-cycles of *Earth* with respect to the *slower* outer (superior) planets or the lap-cycles of the *faster* inner (inferior) planets with respect to *Earth* itself. In both cases, with the reference period of Earth exactly one year, redundant multiplications by unity are unstated, resulting in the standard synodic formulas:

$$\textit{Superior planets } S_s = \frac{T_s}{T_s - 1} \quad \text{(1s)} \qquad\qquad \textit{Inferior planets } S_i = \frac{T_i}{1 - T_i} \quad \text{(1i)}$$

Nevertheless, relation (1) is more useful in the present context, as is relation (2), where, with both the outer period T_1 and intermediate period *S* (= T_2) known, the innermost period T_3 can be obtained from:

$$\textit{Inner period } T_3 = \frac{T_1 \bullet T_2}{T_1 + T_2} \quad (T_1 > T_2 > T_3) \tag{2}$$

Relation (1) permits the restoration of the missing intermediate periods in Table 1b, and allied with relation (2) plus period formulas employing geometric means - relations (4) and (4E) introduced later - all have roles to play in tests on external planetary systems that follow. More immediately, with missing synodic periods supplied and dynamic component incorporated, a standard planetary framework predicated on Peirce's Fibonacci-based approach can now be assembled as follows.

Units of time and measure

Standard years with respect to *unity* and also the Julian year of 365.25 days are applied in the present study, the first for comparison with modern periods in Julian years,[5] and the second for real-time calculations of planetary motion in Part Three utilising the methodology developed by Bretagnon and Simon (1986).[6]

Standard order, positions and titles

Following the order adopted by Pierce, the mean periods of revolution and the mean synodic intervals have been assigned standard position numbers and uniform titles commencing with the first and outermost planet. Thus for the eight-planet Solar System the relative *synodic* period (or lap-cycle) of Planet #2 (Uranus) with respect to that of outermost Planet #1 (Neptune) is *Synodic* 2-1 followed by *Synodic* 3-2 between Planet #3 (Saturn) and Planet #2 (Uranus), and so on, down to *Synodic* 8-7 between innermost planet Mercury (#8) and Planet #7 Venus. Planetary positions interior to Mercury (Intra-Mercurial-Objects, or IMOs) commence at IMO 1 followed by IMO 2, etc., with the intermediate synodic periods, Pierce reduction ratios and later divisors continuing inwards in due order. In this theoretical framework, Earth (with reservations) occupies the *Synodic* 7-6 location between #6 Mars and #7 Venus.

Divisors for the sequential periods of revolution and intermediate synodic intervals

Next, the awkward multiplications by successive reduction factors used by Pierce are replaced by a standard set of divisors applied to the base period alone, a practice already in use for exoplanets. Thus for the eight-planet Solar System the standard integer divisors for the periods of revolution of the planets beginning with the outermost (#1) are: **1, 2, 6, 15, 40, 104, 273, 714**. Divisors for the intermediate mean synodic periods (lap-cycles) are in turn: 1, 4, 9, 25, 64, 169, 441, thus the synodic divisors are all *sequential squares of the Fibonacci Series*.

The complete set of divisors with intermediate synodic divisors shown in brackets is therefore: **1,** (1) **2,** (4) **6,** (9) **15,** (25) **40,** (64) **104,** (169) **273,** (441) **714,** plus (1156) **1870,** (3025) and **4895** for ten-planet systems, etc.

Base periods B1, B2, B3, B4 and B5 for the divisors

Although the period of revolution of the outermost planet (base period B1) is of fundamental importance in Pierce's planetary model, the calculated value for *Synodic* 2-1 is in fact 62,620 days (hereafter base period B2) which exceeds the latter's initial base period of 62,000 days (hereafter, one-off base period P2). Nevertheless, when used as the base period for the divisors, *Synodic* 2-1 yields marginally superior results compared to those obtained with P2. Therefore, where *Synodic* 2-1 differs from B1 a second base period (B2) can be added for further testing. Other bases (B3s) can be approximated by applying the planetary divisors in reverse, *i.e.*, as multipliers of known periods with known locations in otherwise incomplete systems. Where advantageous, the mean value (B4) of multiple B3 products and/or a substitute B5 (yielding least errors) may also be applied at the expense of further complexity.

Resonant triples between planets [RZT]

Resonant triples between planets are included for completeness in Solar System Table 2 and elsewhere. Related to both the twinned Pierce ratios and added divisors, resonant triples are obtained from the bracketing periods of revolution of adjacent planets and the synodic periods in between. Thus, for Neptune and Uranus [1(1)2], Uranus and Saturn [1(2)3], Saturn and Jupiter [2(3)5], etc. Their immediate relevance lies in the fact that the associated divisors are sequential Fibonacci multiples with the central value of each triple providing the multiplication factor. - 1x for the first set: [1(1)2], 2x for the second, thus [2(4)6], 3x for the third [6(9)15], 5x for the fourth [15(25)40], etc.

Fibonacci Periods in days below Mercury

The resulting Pierce P2 planetary framework for a thirteen-planet extension of the Solar System is shown in Table 2a with intermediate positions for the synodic periods and division of modern periods (Base B2/Divisors) included for comparison. The paired resonances from Unity to the Major Sixes (the reverse of the Pierce *reduction* ratios) aid the analyses of exoplanetary systems in Part Two while also bringing to mind ancient methodology, *e.g.*, "Music of the Spheres," which, though not music *per se,* nevertheless appears to have a role in this complex matter. As does the presence of the Fibonacci series below Mercury expressed in days generated by the P2 and the B2 divisors also included in the Table.

PLANETS Synodic	N #	RATIOS (Pierce)	DIVISORS (added)	RESONANCES (to Major 6's)[a]	RES.TRIPLES (to IMO 5)	PERIODS1 P2/Divisors	PERIODS1 T (JYR =365.25)	DISTANCES1 R (Ref. unity/a.u)	Fibonacci[b]	PERIODS2 B2/Divisors
Neptune	1	1/1	1	1 : 1	1(1)2	62,000	169.74675	30.657329		62,620
Synodic	2-1	1/1	1	1 : 1		62,000	169.74675	30.657329		62,620
Uranus	2	1/2	2	Octave #1, 2 : 1	1(2)3	31,000	84.873374	19.312907		31,310
Synodic	3-2	1/2	4	Octave #2, 4 : 2		15,500	42.436687	12.166369		15,655
Saturn	3	2/3	6	Fifth #1 , 6 : 4	2(3)5	10,333	28.291125	9.2846772		10,437
Synodic	4-3	2/3	9	Fifth #2, 9 : 6		6,889	18.860750	7.0855348		6,958
Jupiter	4	3/5	15	Major 6 #1, 15 : 9	3(5)8	4,133	11.316450	5.0404993		4,175
Synodic	5-4	3/5	25	Major 6 #2, 25 : 15		2,480	6.7898700	3.5857029		2,505
M-J Gap	5	5/8	40		5(8)13	1,550	4.2436687	2.6211647		1,566
Synodic	6-5	5/8	64			986	2.6522930	1.9160830		978
Mars	6	8/13	104		8(13)21	596	1.6480267	1.3952204	610	602
Earth/*Syn*	7-6	8/13	169			366	1.0044186	1.0029436	377	371
Venus	7	13/21	273		13(21)34	227	0.6217830	0.7284938	233	229
Synodic	8-7	13/21	441			140	0.3849133	0.5291457	144	142
Mercury	8	21/34	714		21(34)55	87	0.2377405	0.3837681	89	88
Synodic	9-8	(21/34)	1156			54	0.1468397	0.2783315	55	54
IMO 1	9	(34/55)	1870		34(55)89	33	0.0907737	0.2019792	34	33
Synodic	10-9	(34/55)	3025			20	0.0561146	0.1465719	21	21
IMO 2	10	(55/89)	4895		55(89)144	13	0.0346776	0.1063406	13	13
Synodic	11-10	(55/89)	7921			8	0.0214300	0.0771521	8	8
IMO 3	11	(89/144)	12816		89(144)233	5	0.0132449	0.0559800	5	5
Synodic	12-11	(89/144)	20736			3	0.0081861	0.0406179	3	3
IMO 4	12	(144/233)	33552		144(233)377	2	0.0050592	0.0294706	2	2
Synodic	13-12	(144/233)	54289			1	0.0031267	0.0213826	1	1
IMO 5	13	(233/377)	87841			1	0.0019324	0.0155144	1	1

Table2a. The enhanced planetary structure: ratios, divisors, triples, periods in days & years; P2 distances (a.u.).

[a] Octave 2 : 1, Fifth 3 : 2, Major Six 5 : 3. [b] The extension to Planet 13 concludes at Fibonacci number 1.

The Solar System revisited

Table 2b shows the uniform assignments, the twinned Pierce ratios, added divisors, resonant triples and the results generated by Pierce base P2, followed by modern base periods B2 and B1 with the latter in both Julian years and days. Also included with the two sets of data are the calculated synodic periods, Mars-Jupiter geometric mean between the periods of the latter pair, associated synodic positions on either side, and Earth located in the synodic position between Venus and Mars. The 365.25 day period for Earth is substituted in the second set of modern data although the actual synodic period (*Synodic* 7-6) is 335 days, thus less than one year and 366-day period obtained from the P2 ratio and 62,000-day base period. Atypical Venus-Earth and Earth-Mars synodic periods and the Mars-Jupiter synodic cycle are omitted for clarity. The periods in days are rounded; red periods equal exact Fibonacci numbers.

PLANETS Synodics	N #	RATIOS (Pierce)	DIVISOR (added)	RES.TRIPLE [(RZT)]	PERIODS1 P2/Divisors	PERIODS2 Actual/days	MODERN1 B2/Divisors	MODERN2 B1Julian yrs	MODERN1 (Days)	MODERN2 (Days)
Neptune	1	1/1	1		62,000	60,129	171.44429	163.72320	62,620	59,800
Synodic	2-1	1/1	1	1(1)2	62,000	62,672	171.44429	171.44429	62,620	62,620
Uranus	2	1/2	2		31,000	30,687	85.722145	83.747407	31,310	30,589
Synodic	3-2	1/2	4	1(2)3	15,500	16,658	42.861072	45.360219	15,655	16,568
Saturn	3	2/3	6		10,333	10,759	28.574048	29.423519	10,437	10,747
Synodic	4-3	2/3	9	2(3)5	6,889	7,255	19.049366	19.858872	6,958	7,253
Jupiter	4	3/5	15		4,133	4,333	11.429619	11.856525	4,175	4,331
Synodic	5-4	3/5	25	3(5)8	2,480	2,867	6.8577717	7.8476788	2,505	2,866
M-J Gap	5	5/8	40		1,550	1,725	4.2861072	4.7221497	1,556	1,725
Synodic	6-5	5/8	64	5(8)13	986	1,142	2.6788170	3.1255291	978	1,142
Mars	6	8/13	104		596	687	1.6485028	1.8807111	602	687
Earth/*Syn*	7-6	8/13	169	8(13)21	366	335	1.0144633	1.0000000	371	365
Venus	7	13/21	273		227	225	0.6280011	0.6151826	229	225
Synodic	8-7	13/21	441	13(21)34	140	145	0.3887626	0.3958008	142	145
Mercury	8	21/34	714		87	88	0.2401186	0.2408445	88	88
Synodic	9-8	21/34	1,156	21(34)55	54	55	54.169573	54.689759	54	55
IMO 1	9	34/55	1,870		33	34	33.486645	33.723773	33	34
Synodic	10-9	34/55	3,025	34(55)89	20	21	20.700835	20.860438	21	21
IMO 2	10	55/89	4,895		13	13	12.792651	12.888208	13	13
Synodic	11-10	55/89	7,921	55(89)144	8	8	7.9055709	7.9663542	8	8
IMO 3	11	89/144	12,816		5	5	4.8860820	4.9232407	5	5
Synodic	12-11	89/144	20,736	89(144)233	3	3	3.0198701	3.0427860	3	3
IMO 4	12	144/233	33,552		2	2	1.8663570	1.8805320	2	2
Synodic	13-12	144/233	54,289	144(233)377	1	1	1.1534570	1.1622358	1	1
IMO 5	13	233/377	87,841		1	1	0.7128793	0.7183005	1	1

Table 2b. The complete framework and the Solar System. Positions, ratios, divisors and Base periods P2, B1, B2.

Solar System Periods, Pierce Ratios and Divisors below Mercury

Originally the inner region was limited to *Synodic* 9-8 and Planet 9 (IMO 1) to accommodate Pierce's unused inner reduction ratio of 21/55. Accordingly, relation (2) was applied twice, firstly to the mean periods of *Synodic* 8-7 and Mercury resulting in 54.689759 days for *Synodic* 9-8, and then once again to the latter period and that of Mercury to obtain 33.723773 days for Planet #9. However, the last two rounded periods are clearly sequential Fibonacci numbers 55 and 34, an occurrence that allied with the previous sequential pair of periods (145 and 88 days versus Fibonacci 144 and 89) provided the impetus to extend the range as far as Planet 13 (IMO 5) in Tables 2a and 2b.

Regarding the present location of Earth *near* the Mars - Venus *synodic* position, the calculated synodic period, *i.e.*, *Synodic* 7-6 = 335 days represents an enigma since it is neither 366-days as required by the divisors, nor it is close to the actual 365.25 days (Julian) and other variants for the year. Although perhaps masked by a possible outward shift by Mars, this still does little to explain the obvious Fibonacci/*Phi* ratio exhibited by the Venus-Earth periods of revolution expressed in years. In more detail, using modern values for these two adjacent planets the mean periods are 0.61518257 : 1, whereas the reciprocals of *Phi* and Earth (Unity) are 0.61803398875 : 1. Furthermore, there is also the well-known 5 : 8 ratio between the two planets and associated 5 : 8 : 13 *Fibonacci* resonant triple, *i.e.,* 5 synodic periods of Venus in 8 years with 13 corresponding periods of revolution for this planet. All of which, in addition to the above Fibonacci data from Mercury through IMO5, leads logically enough to the following major expansions.

The Pierce planetary framework, the *Phi*-series, and the structure of the Solar System

It is abundantly clear from Table 1b that the final Pierce reduction ratios are successive twinned members of the Fibonacci series, albeit one position removed between the numerators and denominators. Nevertheless, despite the title of Pierce's original publication[1] and obvious nature of the ratios applied by the latter, the Fibonacci series and related Golden Ratio *Phi* (ϕ) :

$$Phi\ (\phi) = \surd(5/4) + \tfrac{1}{2} = 1.618033988749895 \qquad (3)$$

– are nowhere stressed by Peirce or Agassiz, although this constant clearly plays a major role in the proposed model. This is all the more apparent when it is recalled that the golden *section* is defined as the division of a line such that the proportion of the smaller section to the larger is identical to the proportion of the larger section to the whole. Whereas the golden *ratio* can be defined as the limiting value of the ratios of adjacent Fibonacci numbers. It is also clear in the present astronomical context that moving inwards, the limiting value of the inverse alternate Fibonacci ratios applied by Peirce will be ϕ^{-2} (0.38196601125) with reciprocal limit the outward multiplier ϕ^{2} (2.61803398875). Furthermore, after the inclusion of the ratios for the intermediate periods between planets the limiting value is ϕ^{-1} (0.61803398875) with a reciprocal limit and a corresponding multiplier of ϕ^{1} (1.61803398875), which is *Phi* itself.

***The Phi*-series in astronomical context** *(Periods T, S years, Distance R, Velocity Vi and Vr relative to unity)*

As it so happens, apart from filling the intermediate gaps introduced by Pierce, relation (1) – the general synodic formula – is already present with one central exception among the four constants just mentioned, *i.e.*, ϕ^{-2}, ϕ^{-1}, ϕ^{1} and ϕ^{2}. In short, combined with the calibration and the unification provided by the mean period of Earth (ϕ^{0} = 1 year) the latter become sequential mean periods *in years* generated by the *Phi*-series ϕ^{x} for successive integer exponents x = -2, -1, 0, 1, 2 in the present context. Moreover, with the addition of the next lower integer and also continued outward extensions, integer exponents -3 through 7 generate a complete planetary framework from Mercury to Saturn with all synodic periods included. Beyond this outer region correlation with the solar system parameters begins to diminish, but nevertheless, for the stipulated range the inter-related parameters are as shown in Table 3. Here, following ancient practice it is helpful to include the *inverse velocity* *Vi*, *e.g.*, $Vi^{2} = R$, $Vi^{3} = T$, $Vi^{-1} = Vr$ (best remembered by the Triple interval [$3^{0}3^{1}3^{2}3^{3}$ = 1, 3, 9, 27][7] which also pertains to Saturn at perihelion) with the frame of reference (unity) provided by the mean heliocentric distance (*R*) in a.u, mean period of revolution (*T*) in years and mean orbital velocity (*Vr*) of Earth. Thus in the same sense [1,1,1,1], hence the assignment of the cube to planet Earth and *tetradic* point-line-area-volume analogy applied to planetary motion. The last three modern periods in days (in red) owe their origins to relation (2) and a 33.0225-day period for IMO 1 by Leverrier (1875).[8]

PLANETS N Synodics #	MODERN *T* (Julian Years)	ϕ^{x} x	= *Phi*-series *T* (Years)	*Phi*-series (*R*) Distance (a.u.)	*Phi*-series (*Vi*) Inverse Velocity	*Phi*-series (*Vr*) Velocity (Ref.1)	MODERN *T* (Days)
Saturn 3	29.4235194	7	29.03444185	9.446602789	3.073532624	0.325358512	10746.9404
Synodic 4-3	19.8588721	6	17.94427191	6.854101966	2.618033989	0.381966011	7253.45303
Jupiter 4	11.8565250	5	11.09016994	4.973080251	2.230040414	0.448422366	4330.59576
Synodic 5-4	7.84767877	4	6.854101966	3.608281187	1.899547627	0.526441130	2866.36470
M-J Gap 5	4.72214968	3	4.236067977	2.618033989	1.618033989	0.618033989	1724.76517
Synodic 6-5	3.12552908	2	2.618033989	1.899547627	1.378240772	0.725562630	1141.59949
Mars 6	1.88071105	1	1.618033989	1.378240772	1.173984997	0.851799642	686.929711
Earth/*Syn* 7-6	0.91422728	0	1.000000000	1.000000000	1.000000000	1.000000000	365.25(JYR)
Venus 7	0.61518257	-1	0.618033989	0.725562630	0.851799642	1.173984997	224.695433
Synodic 8-7	0.39580075	-2	0.381966011	0.526441130	0.725562630	1.378240772	144.566223
Mercury 8	0.24084445	-3	0.236067978	0.381966011	0.618033989	1.618033989	87.9684354
Synodic 9-8	0.14474748	-4	0.145898034	0.277140264	0.526441130	1.899547626	54.6897591
IMO 1 9	0.09041068	-5	0.076806725	0.201082619	0.448422366	2.230040414	33.0225000
Synodic 10-9	(0.0556507)	-6	0.055728090	0.145898034	0.381966011	2.618033989	20.3264209
IMO2 10	(0.0344447)	-7	0.040434219	0.105858161	0.325358512	3.073532624	12.5818709

Table 3. Modern periods *T*, *S*, *Phi*-series, exponents (x), *T*, *R*, Velocity *Vi* (Inverse) and *Vr* (relative to unity).

Returning to the present, notwithstanding the Mars-Jupiter Gap and anomalous location of Earth *between* Mars and Venus, the *Phi*-series planetary framework outlined above includes the following properties and relations:

Heliocentric properties of the *Phi*-series with respect to unity in the Solar System

1. For any three successive *Phi*-series periods, the middle period is the *product* of the periods on either side *divided* by their *difference*. Thus, in the same astronomical context, the general synodic formula, relation (1)

2. If two upper adjacent *Phi*-series periods are known, the third and lower period can be obtained from the *product of the two adjacent periods divided by their sum*. Thus (in addition to relation 1), synodic relation (2)

3. The underlying constant of the *Phi*-series planetary model is ***Phi*** ($\phi^1 = ½ \sqrt{5} + ½ = 1.618033988749895$), the limiting value of successive ratios of the Fibonacci series: 1,1,2,3,5,8,13,21,34,55,89,144,233,377, 610,... (3)

4. For any three successive *Phi*-series periods, the middle period is the *geometric mean* of the two periods on either side, as are the means from positions ± 2, ± 3, etc. Extended geometric means, relations (4), (4E) & (4F)

5. For every *Phi*-series period *except that of Earth* there exists a corresponding Lucas series integer period (_ 3, 4, 7,11,18,29,47,76,123,199,... years) generated by the *alternating Phi*-Lucas relation: $(T, S) = | \phi^x \pm \phi^{-x} |$ Periods of revolution: $T = | \phi^x - \phi^{-x} |$. Intermediate Synodic Periods: $S = | \phi^x + \phi^{-x} |$ *Phi*-Lucas relation (5)

6. Pertaining *to planet EARTH*, the product of the parameters of the planets on either side (Mars and Venus) is *UNITY*, as are *all such Phi-series products*, *i.e.*, periods ± 2, ± 3, etc., both inwards and outwards. Relation (6E)

The limiting *Phi*-series constants in the present astronomical context are:

A: PLANETS: Mean sidereal periods of revolution, mean heliocentric distances, mean orbital velocities:

Phi-series mean *periods* of revolution (*T*) decrease ϕ^{-2} (0.38196601125), Inwards (the Pierce limit) (7)
Phi-series mean *periods* of revolution (*T*) increase ϕ^{2} (2.61803398875), Outwards (8)
Phi-series mean heliocentric *distance* (*R*) increase $\phi^{4/3}$ (1.89954762695), Planets, Outwards (9)
Phi-series *Planet-to-Planet Velocities* (*Vr*) decrease $\phi^{-2/3}$ (0.725562630246), Planets, Outwards (10)

B: SYNODICS: Mean synodic periods, corresponding heliocentric "distances," mean "orbital" velocities:

Phi-series mean *synodic* (*S*) to *Planet* (*T*) decrease ϕ^{-1} (0.61803398875), Inwards (11)
Phi-series mean *synodic* (*S*) to *Planet* (*T*) increase ϕ^{1} (1.61803398875), Outwards (12)
Phi-series mean *synodic* (*R*) to *Planet* (*R*) increase $\phi^{2/3}$ (1.37824077249), Outwards (13)
Phi-series mean *synodic* (*Vi*) to *Planet* (*Vi*) increase $\phi^{1/3}$ (1.17398499671), Outwards (14)
Phi-series mean *synodic* (*Vr*) to *Planet* (*Vr*) decrease $\phi^{-1/3}$ (0.85179964208), Outwards (15)

C: GENERATION:

The *mean periods* of revolution (*T*) and the *mean synodic periods* (*S*) in years from Mercury to Neptune are generated by the *Phi*-Series (ϕ^x) and integer exponents x = -3,-2,-1,0,1,2,3,4,5,6,7,8,9,10 and 11. (16)

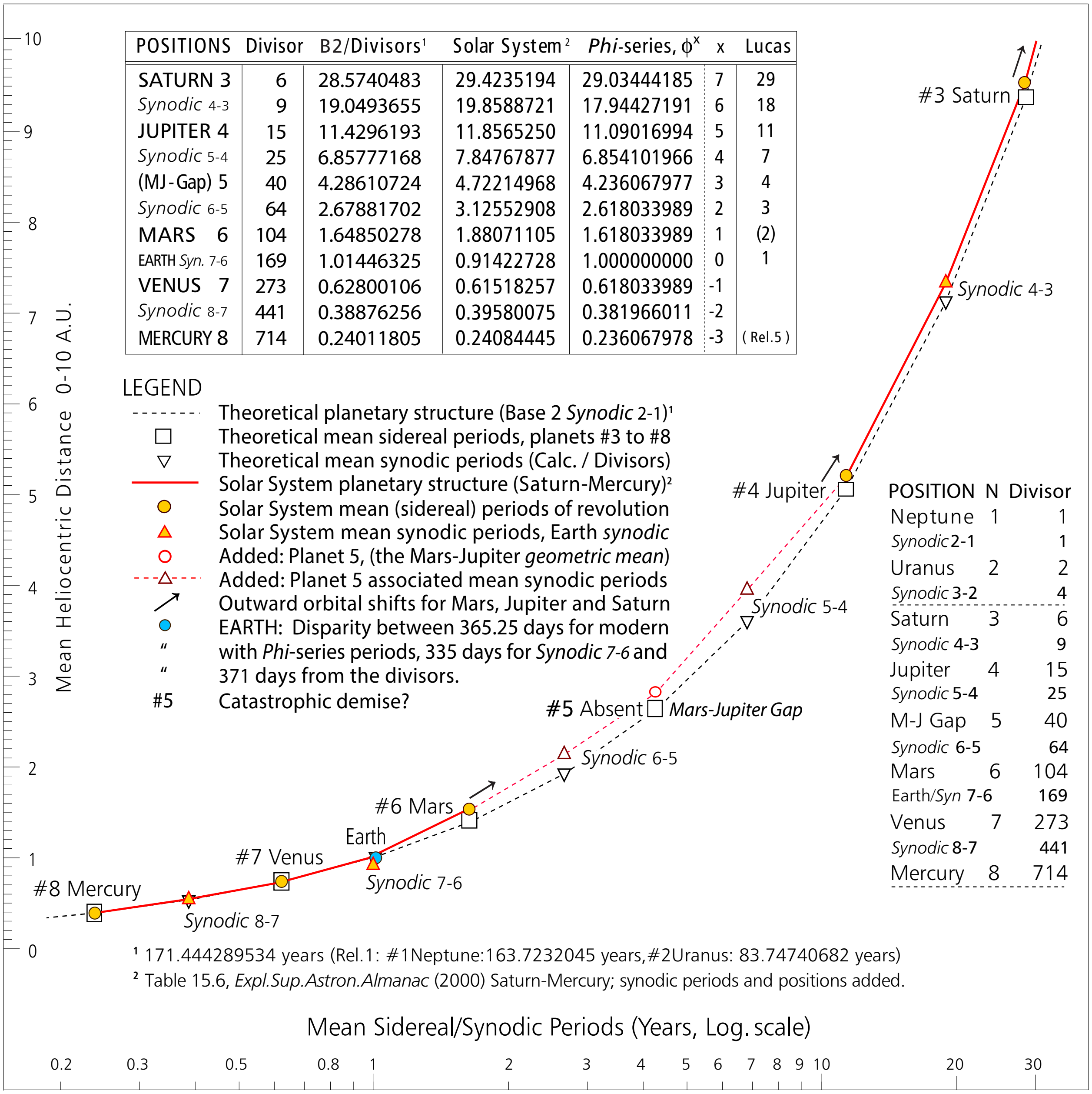
The mean *periods of revolution* are generated by *ODD* exponents, *mean synodic periods* by the *EVEN*. (17)

D: OVERALL PLANETARY FRAMEWORK with increasing departures beyond Saturn (periods *T*, *S* in years). Period divisors, modern values, exponents, Lucas series and *Phi*-series framework are shown in Table 3s. *n.b.*, the *Phi*-series also includes each key *Pheidian* constant as a mean period (*T,S*), a mean distance (*R*) and both velocities (*Vr* & *Vi*) with the latter (0.381966011) at *Synodic* 10-9) not shown. (18)

DIV. (syn)	PLANETS N Synodics #	MODERN *T,S* (Julian years)	ϕ^x exp.	*LUCAS* (years)	*Phi*-series *T,S* (ϕ^x = years)	*Phi*-series (*R*) Distance (a.u.)	*Phi*-series (*Vr*) Velocity (ref.1)	*Phi*-series (*Vi*) Velocity (Inv.)
1	Neptune 1	163.7232045	11	199	199.0050294	34.08599912	0.171282103	5.838321602
(1)	*Synodic* 2-1	171.4442895	10	123	122.9918694	24.73152718	0.201082619	4.973080251
2	Uranus 2	83.7474068	9	76	76.01315562	17.94427191	0.236067977	4.236067978
(4)	*Synodic* 3-2	45.3598213	8	47	46.97871376	13.01969312	0.277140264	3.608281187
6	Saturn 3	29.4235194	7	29	29.03444185	9.446602789	0.325358512	3.073532624
(9)	*Synodic* 4-3	19.8588721	6	18	17.94427191	6.854101966	0.381966011	2.618033989
15	Jupiter 4	11.8565250	5	11	11.09016994	4.973080251	0.448422366	2.230040414
(25)	*Synodic* 5-4	7.84767877	4	7	6.854101966	3.608281187	0.526441130	1.899547627
40	M-J Gap 5	4.72214968	3	4	4.236067977	2.618033989	0.618033989	1.618033989
(64)	*Synodic* 6-5	3.12552908	2	3	2.618033989	1.899547627	0.725562630	1.378240772
104	Mars 6	1.88071105	1	(2)	1.618033989	1.378240772	0.851799642	1.173984997
(169)	Earth/*Syn* 7-6	0.91422728	0	1	1.000000000	1.000000000	1.000000000	1.000000000
273	Venus 7	0.61518257	-1	-	0.618033989	0.725562630	1.173984997	0.851799642
(441)	*Synodic* 8-7	0.39580075	-2	-	0.381966011	0.526441130	1.378240772	0.725562630
714	Mercury 8	0.24084445	-3	(Rel. 5)	0.236067978	0.381966011	1.618033989	0.618033989

Table 3s. Divisors, modern periods (*T* & *S*), *Phi*-series (x) *T, S,* Lucas *T, S,* *Phi*-series *R*, *Velocity Vr* & *Inverse Vi*.

Disparities in the modern Solar System from Mercury through Saturn with emphasis on the Pierce Divisors and insights from the above are shown in Figure 1.



POSITIONS	Divisor	B2/Divisors¹	Solar System²	*Phi*-series, ϕ^x	x	Lucas
SATURN 3	6	28.5740483	29.4235194	29.03444185	7	29
Synodic 4-3	9	19.0493655	19.8588721	17.94427191	6	18
JUPITER 4	15	11.4296193	11.8565250	11.09016994	5	11
Synodic 5-4	25	6.85777168	7.84767877	6.854101966	4	7
(MJ-Gap) 5	40	4.28610724	4.72214968	4.236067977	3	4
Synodic 6-5	64	2.67881702	3.12552908	2.618033989	2	3
MARS 6	104	1.64850278	1.88071105	1.618033989	1	(2)
EARTH *Syn.* 7-6	169	1.01446325	0.91422728	1.000000000	0	1
VENUS 7	273	0.62800106	0.61518257	0.618033989	-1	
Synodic 8-7	441	0.38876256	0.39580075	0.381966011	-2	
MERCURY 8	714	0.24011805	0.24084445	0.236067978	-3	(Rel.5)

Fig. 1. The Pierce planetary framework, Solar System Mercury–Saturn, Mars-Jupiter Gap and the location of Earth.

Similarities and Disparities

Applied to the present Solar System, the *Phi*-series from x = -3 to 7 yields a planetary framework which includes all the intermediate (synodic) periods from Mercury through Saturn plus periods for theoretical planet #5 and both adjacent intermediate synodic intervals. Beyond Saturn correlation diminishes with distance, while the ratios of the of the integral Lucas series increasingly approach *Phi* itself. Whereas, moving inwards, ratios of *the period divisors* also begin to approach the same fundamental constant. Nevertheless, two identical disparities in the Solar System are indicated by all three sequences: (1) the absence of a planet between Jupiter and Mars, and (2), the unexpected presence of a planet between adjacent Mars and Venus, namely Earth. Moreover, in addition to this location there is also a marked difference between the calculated intermediate period for Earth of 335 days and the 365-day year.

In so much as Venus and Earth have the lowest eccentricities among the planets and their periods of revolution are also closest to their *Phi*-series equivalents - with *zero* error for Earth - the position of the latter can be examined in terms of residual effects of the *Phi*-series with relation 6E a possible factor. This, however, is difficult to investigate because of the missing periods between Mars and Jupiter, and also the accepted absence of planets below Mercury.

About the only option remaining pertains to the periods of Jupiter and IMO 1, *i.e.,* the periods corresponding to *Phi*-series exponents +5 and -5 which yield *a product* of exactly 1 year. Whereas in the Solar System the mean period of Jupiter of 11.85652502 Julian years and that of IMO 1 (0.09035592 years) yields a product of 1.071307238, with the replacement of IMO 1 by the mean synodic month resulting in 0.9586044 years. *Unity* does, however result from a period of 30.8058220 days from *the reciprocal* of Jupiter's mean period, a concept which owes its origins to Friberg's approach to AO 6484, a Babylonian mathematical text concerned with the number 0;59,15,33,20 and its reciprocal 1; 00,45.[9] The product is necessarily *unity* with a *sum* of 2; 0,0,33,20 and 1; 0,0,16,40 for the half.[9] Which, albeit radical shifts in both time and place, can be considered in terms of *elliptical parameters for the orbit of Earth*. This is an unexpected bi-product of a reappraisal of the 1964 analyses by A. Aaboe[10] of a possible *daily increment* of 0; 0,1,32,42,13,20° (0.000480109739369) for the velocity of the "Sun" in BM 37089, a Babylonian lunar fragment.

The relevance of the latter is that the value 0;59,15,33,20°/Day can be shown to be inherent in data in Aaboe's study which corresponds to a period of exactly 364.5 years. This value is shown below in the last column of Table 1A from an expansion of Aaboe's analyses incorporating a Babylonian System B *varying velocity* function for planet *Earth:*

> ". . . Although Aaboe surmises that the original table may have supplied daily longitudes for a complete year, he gives a partial restoration since neither the maximum nor the minimum values are present. It is, however, sufficient to give Aaboe's daily longitudes and differences for lines 5 through - 5 plus added corresponding lengths of the year in days to show that line 0 is the closest to the Sidereal year:

Line #	Col. II (Longitude°)	Δ Col. II (Daily velocity°)	*T* (years, added)
Line -5	[8;51,51,51,6,40]	0;59,17,17,2,13,20	364.32289859
Line -4	[9;51,6,40]	<u>0;59,15,33,20</u>	<u>364.5</u>
Line -3	[10;50,20,29,37,46,40]	0;59,13,49,37,46,40	364.67727367
Line -2	[11;49,32,35,33,20]	0;59,12,5,55,33,20	364.85471987
Line -1	[12;48,42,57,46,40]	0;59,10,22,13,20	365.03233883
<u>Line 0</u>	[13;47,51,36,17,46,40]	<u>0;59,8,38,31,6,40</u>	<u>365.21013081</u>
Line 1	[14];46,58,[31,6,40]	0;59,6,54,48,53,20	365.38809607
Line 2	15;46,[3, 42,13,20]	0;59,5,11,6,40	365.56623485
Line 3	16;45,7,[9,37,46,40]	0;59,3,27,24,26,40	365.74454742
Line 4	17;44,8,[53,20]	0;59,1,43,42,13,20	365.92303402
Line 5	18;43,[8,53,20]	0;59.	366.10169492

Table 1A. Daily solar positions and velocities with periods *T* added to Aaboe (1964:32).

> This demonstrates that from a modern perspective the mean daily velocity from line 0 of 0; 59, 8,38,31, 6,40°/d and the 365.21013081-day year are optimum for (*u*) and (*T*) respectively. But not quite. In order to restore the longitudes and velocities for the entire table, the period *T* turns out to be exactly 364 days. Thereafter, with (*d*) given, (*u*) from Line 0, and *T = 364* days, relation (X) is reduced to: (*M, m*) = 0;59,8,38,31,6,40 ± 0;2,37,17,2,13,20 which produces the following six-sexagesimal place values for the apsidal velocities and the daily velocities in between.
>
> Minimum daily velocity (*m*) = 0;56,31,21,28,53,20°/day (abbrev. 0;56,30)
> Mean daily velocity (*u*) = 0;59, 8,38,31, 6, 40°/day (abbrev. 0;59, 9)
> Maximum daily velocity (*M*) = 1; 1,45,55,33,20,00°/day (abbrev. 1; 1,46)
> eccentricity (*e*) = 0.0295589.
>
> The occurrence of 0;59° in line 5 of column 3 suggests choice rather than coincidence and there are other matters of interest in addition." [Excerpt from "Aaboe64 Revisited"].

The above dialogue concludes with an associated ellipse and additional variants which are beyond the scope of the present study. Except to note that Friberg's analysis mentioned earlier is accompanied by a two-part figure for the Babylonian mathematical procedure known as "Completing the Square." The latter, however, in consort with the calculation of the heliocentric distances *R* (by a procedure provisionally named here "Completing the Cube" inherent in Old Babylonian mathematical text VAT 8547) suggests that these procedures ultimately concern the derivation of the parameters of *ellipses* for Earth and the major superior *Planets*. In so much as the eccentricities (*e*) are small (*e.g.*, that of Earth is 0.01670862) the orbits appear to be *almost* circular, which provides an impetus to revisit Babylonian mathematical texts with accompanying "circles" and non-integer numerical values *close to unity or 2*. Thus possible semi-major (*a*) and major axes (2*a*) for Earth/*Sun ellipses, e.g.,* although conceivably with alternate meanings:

> "Fig. 3. 1. 12. MS 3050. An OB round hand tablet with square inscribed in a circle." Friberg (2005:135).[11]
> "Fig. 16. 7. 3. *UET*/67 2 222 *rev*. A square side algorithm using elimination of square factors." Friberg (2006:401) [12]
> "Fig. 16. 7. 4. 1st. Si. 428. Computation of the square side 2;02,02,02,05,05,04." Friberg (2006 :403).[13]

where the first figure appears to be a rough *rectangl*e with diagonals inscribed in an equally rough *ellipse*.

Seeking further enlightenment the inquiry leads to Babylonian planetary and luni-solar parameters, but before this it is necessary to caution the casual reader about prevailing nihilistic views concerning Babylonian astronomy, especially ill-founded claims that the Babylonians had neither a fictive approach to orbital motion nor any planetary model whatsoever. Long overdue additional research shows that nothing could be further from the truth.

But before proceeding, the notation, conventions and additional data in this context are introduced for those who may be unfamiliar with this relatively obscure material, along with standard definitions of astronomical terms, and in particular, luni-solar and planetary parameters in both modern and Babylonian contexts.

Sexagesimal notation, Units, Time, and Motion

Sexagesimal numbers 1 to 59 are separated by commas with equivalent decimal place locations indicated by semi-colons, thus in addition to hours; minutes and seconds, the thirds, fourths, fifths, sixths, sevenths, etc. For example, the Old Babylonian estimate for the square root of 2 rounded at the *third* place is 1;24,51,10[13] with the *exact* value for the Babylonian mean synodic arc of Saturn[13] 12;39,22,30° (12.65625°) with a corresponding mean synodic time of 1;2,6,33,45 years (1.03515625 versus the modern mean synodic period for this planet of 1.035182135… years). Days, degrees, months and "tithis" (thirtieths) are denoted by the superscripts n^d, n^o, n^m, n^r with the predominant Babylonian mean synodic month (MSM) of 29;31,50,8,20 ^{d} (29.5305941358...^{d}) represented by superscript ^{M}.

Next, expanded later, definitions and tools for the present study include the following luni-solar constants:

(1) DAY: *Daily axial rotation* and *daily sidereal motion of Earth* with subdivisions of the 24-hour day for time & angular motion which far exceed modern usage, extending from 360° per day through Large Hours (30°), Hours (15°), Minutes and Seconds, etc., down to 50 seconds of arc (0;00,50°).

(2) MONTH: MEAN SYNODIC MONTH of 29;31,50,8,20 days = 29.5305941358^d with last base-60 pair rounded for convenience. Even so it is still quite accurate; the modern estimate is 29;31,50,*7*,30 days.

(3) YEAR: SIDEREAL YEAR of 12;22,8 Mean synodic months = 365;15,38,17,44,26,40 days (365.2606376886). Although the latter is high compared to the modern estimate of 365.2564^d it is almost certainly selected for convenience. A better estimate for the sidereal year is also available from the accurate Babylonian mean sidereal month of 27;19,18^d and above mean synodic month which generate a year of 365.2564698 days.

(4) METHODOLOGY: Explanations of the fundamental motions involved according to the methods laid out in the Babylonian procedure texts and related data determined from the Babylonian end products, *i.e.*, the Ephemerides. And in addition, the implications of the Earth/Sun duality in the Babylonian context.

(5) Closely associated to (4), the underlying formulas required to assess Babylonian results and procedures. In this case, since Babylonian planetary theory deals to a considerable extent with *synodic* motion, and the latter understanding is also applicable to the lunar component, the computation of synodic cycles, synodic periods and synodic arcs also play a role in the current investigation, the following especially:

Synodic periods and synodic formulas

The synodic period (S) or lap-cycle between two Solar System planets with mean periods of revolution T_1 and T_2 is given by the *general synodic formula for co-orbital bodies* applied earlier to the Pierce data :

$$\text{Synodic period } S = \frac{T_1 \cdot T_2}{T_1 - T_2} \; (T_1 > T_2) \qquad (1)$$

along with the simplified standard synodic formulas for the Superior and Inferior planets:

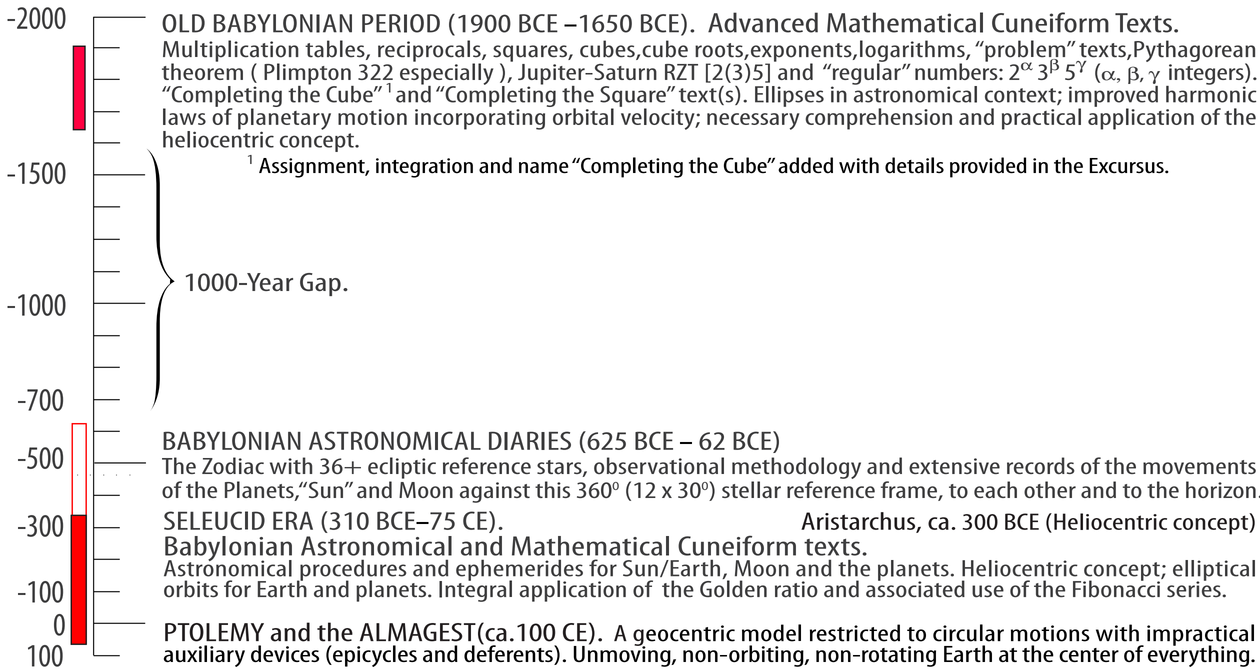
$$\text{Superior planets, } S_s = \frac{T_s}{T_s - 1} \qquad (1s) \qquad\qquad \text{Inferior planets, } S_i = \frac{T_i}{1 - T_i} \qquad (1i)$$

augmented, if required, by synodic relation (2) where periods T_1 and S are known and period T_2 is of interest:

$$\text{Period } T_2 = \frac{S \cdot T_1}{S + T_1} \; (S > T_2) \qquad (2)$$

Synodic relations (1), (2) and modern equivalents all have roles to play in what follows, but relation (1) in full has a further application arising from the inclusion of the mean synodic month in Babylonian planetary theory beyond calendaric considerations. Although obvious, this was either missed or - for whatever reasons - ignored by noted authority Otto Neugebauer. More on this matter later.

As for the relevance of Babylonian astronomy in the presence context, further examination the mathematical cuneiform texts from the Old Babylonian Period (1900 BCE - 1650 BCE),[14] the Babylonian astronomical diaries from 652–62 BCE,[15] details in the Babylonian astronomical "procedure" texts and the resulting *Ephemerides* of the Seleucid Era (310 BCE–75 CE)[16] represent an extensive source of largely misrepresented and/or misunderstood information. Included here are specific parameters with descriptions in the procedure texts concerning their determination, sufficient details, in fact, for *the heliocentric concept* and *refined laws of planetary motion* to be added to the already complex mathematics of the Old Babylonian Period. The acceptance of which is adversely influenced by the time line between the sources and lack of connectivity with the earliest in terms of known astronomical concepts.



Text-Fig 1. Old Babylonian Mathematics, Babylonian Astronomical Diaries and Seleucid Era Astronomy.

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Introduction

Although the roles of synodic relations (1) and (2) with respect to the Solar System periods and synodic cycles are not entirely surprising the two relations are nevertheless both underlying elements of the structure of the *Phi*-series planetary framework. Just how well the Babylonian luni-solar material reflects this is another matter, but assuredly the subject is worthy of further investigation, especially with respect to attested Babylonian periods and velocities for the five planets known in Antiquity. But then again, the *Fibonacci, Lucas* and the *Phi*-series are all considered to be relatively recent in both origins and understanding, hence the following introduction to the historical side of the matter.

I. The Fibonacci and the Lucas series in early times.

Although the first of these two elementary series is still credited to Fibonacci (*ca.*1175–1240 CE) and likewise the second to Francois Lucas (1842–1891), as Thompson pointed out over a century ago,[17] it is unlikely that the former would have escaped the attention of Greek philosophers or even earlier inquiring minds. Furthermore, this same argument applies equally (if not more so) to the latter series (1,3,4,7,11,18,29,47) since it is simply the *next additive sequence* after the *Fibonacci, i.e.*, *1,1,**2,3**,5*,... is followed by: *1,**3,4**,7*,... (the *Lucas),* then: *1,**4,5**,9*,.. and *1,**5,6**,11*,... etc., all with the same limiting ratio (ϕ) between adjacent pairs. The last mentioned (provisionally the *Penta* series 1,***5*, *6***,11, 17,**28**,..) also includes the first two perfect numbers **6** and **28** (numbers equal to the sum of their own parts). And eventually, the convenient approximation for the Golden ratio of 809/500 = 1618/1000, thus **1.618** (1;37,4,48).

II. Babylonian Jupiter/Saturn mean synodic arcs; the *Phi*-series and the Golden ratio

Both historically and in astronomical terms, the ratio **5** : **6** is known to play an underlying role in the location of the extremal synodic arcs for Jupiter[18] and Saturn[19] in the Babylonian astronomical cuneiform texts of the Seleucid Era (310 BCE–75 CE). Furthermore, despite current dismissive views on this subject, another point of relevance is found in the Babylonian estimates for the *sidereal* periods of revolution for Jupiter (11;51,40 = 11.86111* years) and Saturn (29;26,40 = 29.444* years) which provide the basis for the mean synodic arcs (*u*) according to Babylonian System B. In particular, it is *the ratios* of these synodic arcs - 33;8,45° (33.1458333*) for Jupiter[20] and 12;39,22,30° (12.65625) for Saturn[21] - which are of immediate interest, since:

$$\frac{\text{Saturn }(u)}{\text{Jupiter }(u)} = \frac{12.65625}{33.14583333^*} = 0.38183534 \textit{ versus } \phi^{-2} = 0.38196601125, \text{ the Pierce Limit, } \textit{Phi}\text{-series relation} \qquad (7)_{S/J}$$

$$\frac{\text{Jupiter }(u)}{\text{Saturn }(u)} = \frac{33.14583333^*}{12.65625} = 2.61893004 \textit{ versus } \phi^{2} = 2.61803398875, \text{ Planet-to-Planet } \textit{Phi}\text{-series relation} \qquad (8)_{J/S}$$

whereas the difference between the two mean synodic arcs, *i.e.*, Jupiter (*u*) -Saturn (*u*) = 20;29,22,30° (20.48958333*) not only provides the arc for the difference cycle *SD*1 between the two planets (Synodic 4-3 in the Peirce framework) but also two further inter-related ratios of similar interest:

$$\frac{\text{Jupiter }(u)}{SD1(u)} = \frac{33.14583333^*}{20.48958333^*} = 1.61769192 \textit{ versus } \phi = 1.61803398875, \text{ Planet-to-Synodic } \textit{Phi}\text{-series relation} \qquad (12)_{J/D}$$

(1.61764706 = Fibonacci 55/34)

$$\frac{SD1(u)}{\text{Saturn }(u)} = \frac{20.48968333^*}{12.65625} = 1.61893004 \textit{ versus } \phi = 1.61803398875, \text{ Synodic-to-Planet } \textit{Phi}\text{-series relation} \qquad (12)_{D/S}$$

III. Babylonian Jupiter/Saturn mean synodic arcs and the Fibonacci series

In addition to this pair of mean synodic arcs, System A′ for Jupiter[22] features an intermediate arc of 33;45° (33.75° = u_2) as opposed to (*u*), the mean synodic arc of 33;8,45.° Retaining Saturn's mean synodic arc of 12;39,22,30°. but using 33.75° for Jupiter and new difference arc *SD*1′= 21;5,37,30° (21.09375) the divisions for the new arcs now yield the following familiar Fibonacci ratios which suggests the previous relations are *unlikely to be coincidental or unknown*;

$$\frac{\text{Jupiter }(u_2)}{SD1(u)'} = \frac{33.75}{21.09375} = 1.6 \text{ (1;36). Fibonacci ratio (8/5)} \qquad (12)_{J/DF}$$

$$\frac{SD1(u)'}{\text{Saturn }(u)} = \frac{21.09375}{12.65625} = 1.666^* \text{ (1;40). Fibonacci ratio (5/3)} \qquad (12)_{D/SF}$$

$$\frac{\text{Jupiter }(u_2)}{\text{Saturn}(u)} = \frac{33.75}{12.65625} = 2.666^* \text{ (2;40). Fibonacci ratio (8/3)} \qquad (8)_{J/SF}$$

IV. Jupiter and Saturn mean value ratios for Babylonian Systems A and B

Remaining with Jupiter and Saturn, there is a major difference between the two primary methods for dealing with *varying* synodic motion (Systems A and B) with the two-velocity configurations of System A using a minimum arc (w) and a maximum arc (W) sensibly understood to be *apsidal* velocities with *pheidian* elements in an associated ***5 : 6*** ratio. For Jupiter the minimum and maximum synodic velocities (or apsidal arcs) are 30° and 36°,[23] whereas for Saturn the minimum (w) and the maximum (W) have a marked difference in the number of sexagesimal places, *i.e.*, (w) = 11;43,7,30° (11.71875), and (W) = 14;3,45° (14.0625).[24]

On further examination, however, it seems possible that the latter set may have originated from the former since the seemingly more accurate apsidal synodic arcs for Saturn can be derived from the Jupiter data by s*imple division*, *i.e.,* 36°/2.56 = 14;3,45° (14.0625) and 30°/2.56 = 11;43,7,30° (11.71875). Plus one further point; the common divisor is also the *square* of Fibonacci ratio 8/5 ($1.6^2 = 2.56$), thus a practical reduction factor for the periods of revolution of these two adjacent major planets in keeping with the *Fifths* and the *Sixths* of the final Pierce framework.

It is, however, more complicated than this, for even though Jupiter's new value from System A' (u_2 = 33;45° as used in relations $(12)_{J/DF}$ though $(8)_{J/SF}$) resulted in three Fibonacci ratios using this constant, it is not the actual mean value for Jupiter, which is ½ ($W + w$) = (u_3) = 33°. Whereas the mean value from Saturn's System A is in turn found to be ½ (14;3,45°+11;43,7,30°) = (u_4) = 12;53,26,15° (12.890625) with the ratio between the two new mean values now:

$$\frac{\text{Jupiter } (u_3)}{\text{Saturn}(u_4)} = \frac{33}{12.890625} = 2.56 = 1.6^2.\ \text{Fibonacci ratio } (8/5)^2 \qquad (8)_{J/SF2}$$

while the ratio between Jupiter (u_2) and Saturn (u_4) is:

$$\frac{\text{Jupiter } (u_2)}{\text{Saturn}(u_4)} = \frac{33.75}{12.890625} = 2.6181818^* = \text{Fibonacci ratio } 144/55 \qquad (8)_{J/SF3}$$

with a reciprocal of: $$\frac{\text{Saturn } (u_4)}{\text{Jupiter}(u_2)} = \frac{12.890625}{33.75} = 0.3819444^* = \text{Fibonacci ratio } 55/144 \qquad (7)_{S/JF2}$$

and a corresponding ideal growth angle (360°• 0.3819444*) of 137.5°.

Lastly, with a new difference arc *SD*1 of (33.75-12.890625) = 20.859375 (u)", relation $(12)_{J/DF}$ now becomes:

$$\frac{\text{Jupiter } (u_2)}{SD1(u)''} = \frac{33.75}{20.859375} = 1.6179775281 = \text{Fibonacci ratio } (144/89) \qquad (12)_{J/DF2}$$

At which point Babylonian astronomy in general and the origins of these mean synodic arcs in particular begin to assume an unexpected level of importance despite almost universal dismissal of Babylonian methodology at the present time. For this reason the Babylonian observational reference frames and resulting luni-solar parameters in particular offer a minor introduction to the optional excursus at the end of Part 1.

V. Babylonian luni-solar parameters and *Phi*-series/synodic relations (1) and (2)

The inclusion of luni-solar parameters in the present context gives rise to the following *added* abbreviations, names, descriptions and periods in base-60 with decimal equivalents. All *bar* the tropical month and the tropical year were gleaned from leading authority O. Neugebauer's barely readable sexagesimal analyses rendered less understandable by the latter's non-model approach to Babylonian planetary theory. Because of these problems the following tables are largely *prior analytics* initially limited to mean values for synodic relation (1) subroutines applied in Table AP2

Abbr.	Astronomical Names and Standard Descriptions	Babylonian periods	Decimal days
MSM:	Mean Synodic month (new moon to new moon).	29;31,50,8,20 (rounded)	29.530594136 d
MSID:	Mean Sidereal month (fixed star to fixed star).	27;19,18 (rounded)	27.321666667 d
MTROP:	Tropical month (equinox to equinox; text, calc., added).	27;19,17,45 (rounded)	27.321574074 d
MAN:	Anomalistic month (perigee to perigee).	27;33,20 (unrounded)	27.555555555 d
MDRA:	Draconic month (node to node). ACT.	27;12,44 (rounded)	27.212222222 d
MDRA2:	Draconic month (node to node), ACT. calc.	27;12,43,59,40 (rounded)	27.212220679 d
SYR:	Sidereal year (fixed star to fixed star). calc. 12;22,8•MSM	365;15,38,17,44,26,40.	365.26063769 d
SYR2	Sidereal year (fixed star to fixed star). calc. (MSM : MSID).	365;15,23,17,30.	365.25646991 d
TYRB:	Tropical year (equinox to equinox; (Bab. ACT 210, Sect.3)	365;14,44,51	365.24579167 d
AYR:	Anomalous year (perihelion to perihelion (calc., added).	365;15,34,18,22,58,51,	365.25952955 d
EYC:	Eclipse cycle (lunar node to lunar node) (text/mult/calc.).	(5458/465)•MSM.	346.61931784 d
AYC:	Anomalistic cycle (text/mult/calc; added)	(251/18)•MSM.	411.78772933 d

Table AP1. Astronomical terms, Babylonian mean luni-solar periods and decimal equivalents I

Next, the role played by the two primary *Phi*-series/synodic relations in the present context should also be noted:

$$\textit{Synodic period } S_2 = \frac{T_1 \bullet T_3}{T_1 - T_3} \; (T_1 > S_2 > T_3) \quad (1) \qquad \textit{Inner period } T_3 = \frac{T_1 \bullet T_2}{T_1 + T_2} \; (T_1 > T_2 > T_3) \quad (2)$$

Significantly, the mean synodic month (MSM = T_1) and the tropical month (MTROP = T_1 *and* T_2) also play a role in the comparisons between other Babylonian luni-solar cycles, mean luni-solar periods and the modern values with variants of *Phi*-series/synodic relation (1) predominating in Table AP2.

#	Cycles and/or Periods	Subroutine (T_1>T_2>T_3)	Mean periods	Modern equivalents/; (sources)	Relations (1x)
[1]	Eclipse cycle (EYC)	(MSM : MDRA)	= 346.619576 days	(Modern: 346.620107 days; (calc.)	(1e)
[2]	Anomalistic cycle (AYC)	(MSM : MAN)	= 411.780405 days	(Modern: 411.783870 days; (calc.)	(1a)
[3]	Nodal cycle (days)	(MTROP : MDRA)	= 6797.54400 days	(Modern: 6798.26051 days; Tables: 6798)	(1n)
--	Nodal cycle (years)	(MTROP : MDRA)	= 18.6108756 years	(Modern: 18.6128373 years:(calc.,)	(1n)
[4]	Lunar perigee	(MAN : MTROP)	= 3231.88186 days	(Modern: 3231.56072 days; Tables: 3232)	(1p)
[5]	Sidereal year (SYR. MYR)	(MSM •12;22,8)	= 365.260637 days	(Expressed in mean synodic months = MYR)	--
[6]	Sidereal year (SYR2,calc.)	(MSM : MSID)	= 365.256469 days	(Modern: 365.256365 days; Tables)	(1s)
[7]	Tropical year (TYR, calc.)	(MSM : MTROP)	= 365.244059 days	(Modern: 365.242189 days; Tables)	(1t)
[8]	Tropical year (TYR, text)	(TYRB: 18-yr pd	= 365.245792 days	365;14,44,51 days (ACT 210, Sec. 3)	--
[9]	Anomalistic year (AYR)	MSM•(360°/(u°)	= 365.259529 days	(Modern: 365.259641 days; Tables)	u = 29; 6,19,20°
[10]	SAROS, 19 EYC or	(223 MSM, calc.)	= 6585.32249 days	(Modern: 6585.32163 days; Tables)	--

Table AP2. Astronomical terms, Babylonian luni-solar cycles and decimal equivalents II.

For example, although the slightly too large yet practical sidereal year SYR (#[5]) is12;22,8 mean synodic months or 365.260637 days, the more accurate value (SYR 2 = 365.256469 days) is readily available by way of synodic relation (1) utilizing the Babylonian mean synodic month (MSM = 29;31,50,8,20 days) and the mean sidereal month (MSID = 27;19,18 days):

$$\text{SYR2} = \frac{\text{MSM} \bullet \text{MSID}}{\text{MSM} + \text{MSID}} = 365.256469811878 \text{ (365;15,23,17,28,45,43 days)} \qquad \text{(1Syr2)}$$

VI. The Tropical month from Babylonian luni-solar parameters

Also noteworthy are the extended Babylonian luni-solar cycles, especially those stated in eight lines of lunar text No. ACT 210, Section 3.[23] Although rarely recognized as such, they include one of the more contentious issues likely to arise in this context, *i.e.*, *presence of the Tropical month and the Tropical year* in Babylonian astronomy. The latter ([8] in Table AP2) occurs as "1,49,45,19,20 days of 18 years of the moon,"[25] yielding 354;14,44,51 days, which is superior to that used by Claudius Ptolemy (365;14,48 = 365.24666* days). More helpful, however, the presence of a tropical year supplies the means for determining a theoretical length for the *Tropical month* in Babylonian astronomy.

Applying a value for the Tropical year (TYRB) of 365;14,44,51days and mean synodic month (MSM) of 29;31,50,8,20 days, an estimate for the tropical month (MTROP) is available from synodic relation (2) *i.e.*, subroutine TYRB : MSM:

$$\text{MTROP} = \frac{\text{TYRB} \bullet \text{MSM}}{\text{TYRB} + \text{MSM}} = 27.32160692 \text{ (27;19,17,47,5, ... days)} \qquad \text{(2tr)}$$

which rounds conveniently to 27;19,17,45 days and the even more convenient Babylonian estimate of 29;19,17,40 days for (perhaps) ACT 210 Section 3. The assignment of 365;14,44,51^d for a Babylonian tropical year was previously proposed by Hartner in an erudite discussion concerning *the tropical year and precession* which ended as follows[26]

> The inevitable conclusion to be drawn from the preceding demonstrations is, that in Babylonia under Achaernenian rule at the latest in 503 B.C., a clear distinction is made between the length of the tropical year: A = 365;14,48,33,37^d (possibly already then found exchangeable in practice with Ar = *365;14,44.51*^{d}) and that of the sidereal year as underlying System B: PB ′ = 365;15:34, 18,1... ^{d} (italics supplied)
>
> Willi Hartner, "The Young Avestan and Babylonian Calendars and the Antecedents of Precession." *JHA*, X,1979:1–22.

VII. Precession and the Babylonian Sidereal/Tropical years

Thus once again *Phi*-series/General synodic relation (2) is indicated, albeit with respect to mean values, whereas although the standard sidereal year [5] and tropical year [8] are both on the high side, their difference nevertheless yields a Seleucid Era value (perhaps known, perhaps not) for annual precession of 0;0,52,40,41,...° and 24,602 years for the full cycle.

VIII The Anomalistic year

Unlike the derivations based on synodic relations *the anomalistic year* can be obtained from the *mean sidereal arc of Earth* (29;6,19,20°) *per mean synodic month* of 29;31,50,8,20^d. This ratio yields a daily velocity of 0;59,8,9,43,22,7, ...°

$$\text{Mean daily velocity of Earth} = \frac{29;6,19,20^{\circ}}{29;31,50,8,20^{d}} = 0;59,8,9,43,\mathit{22,7}, \quad (0;59,8,9,43,20 \text{ rounded} = u') \qquad (3m)$$

for a corresponding year of 365;15,34,18,22,58,51,40^d or 365.2595295... days. The modern estimate is 365.259641. . Or more simplistically, the amount moved by Earth along its orbit from one full-moon to the next. Thus, from ratio (3u) the mean period of Earth in mean synodic months is :

$$\text{Period of revolution of Earth} = \frac{360^{\circ}}{u'} = 12;22,7,51,53,40,\ldots \text{ mean synodic months} = 365.25952955 \text{ days} \qquad (3u)$$

IX. Multiple luni-solar extensions from *Phi*-series/synodic relation (1)

The simplicity of this relation permits similar derivations for the Draconic, Anomalistic and Nodal Cycles. The first pair include the mean synodic month (MSM) whereas the last cycle uses the Draconic (MDRA) and Tropical (MTROP) months:

$$\text{Draconic Cycle} = \frac{\text{MSM} \cdot \text{MDRA}}{\text{MSM - MDRA}} = 346.6195761217 \ldots \text{ days} \qquad (1dc)$$

$$\text{Anomalistic Cycle} = \frac{\text{MSM} \cdot \text{MAN}}{\text{MSM - MAN}} = 411.7805352634 \ldots \text{ days} \qquad (1ac)$$

$$\text{Nodal Cycle} = \frac{\text{MTROP} \cdot \text{MDRA}}{\text{MTROP - MDRA}} = 18.6101191842.. \text{ years} \qquad (1nc)$$

Here the nodal cycle is of potential interest in view of its association with lunar standstills in the first place and the apparent trouble the ancients took to delineate this phenomenon in the second, *e.g.* Chaco Canyon in the United States, Stonehenge in England and Callanish in Scotland.[27]

At this point Babylonian astronomy in general and the origins of these mean synodic arcs in particular begin to assume an unexpected level of importance despite almost universal dismissal of Babylonian methodology at the present time. For this reason it appears necessary to to offer an optional **excursus** after the Bibliography for Part V to explain the statements in Text-Fig 1concerning advanced knowledge of astronomy in the Old Babylonian period and other matters of concern.

PART ONE: CLOSING REMARKS

Rejections: (1) Expansions of the Laws of planetary motion; (2) Benjamin Pierce's planetary framework

Starting with Galileo and the velocity expansions of the laws of planetary motion described in the Excursus, the concern here is that while the present writer was merely a tertiary restorer, and as such did not expect much in the way of applause, it seemed a reasonable assumption that the extended version $T^2 = R^3 = Vi$,[6] $R = Vi^2$ would at least take its place next to Kepler's twin parameter format $R^3 = T^2$. And further, that variants of the former would simplify routine tasks, *e.g.,* the calculation of angular momentum *L*, Table 1 mean velocities and the like. But this did not come to pass, and so it has remained ever since. On the other hand, modern science appears to have been able to function without such expansions, though not necessarily as well, it is suggested, had these velocity components also been incorporated.

But the real problem is not this historical item *per se*, but rather, that the same process and rapid dismissal was also applied to Benjamin Peirce's Fibonacci-based planetary framework with no replacement or improved version to take its place. And oddly, because of this situation which has remained unaddressed, humankind is now avidly searching for external planetary systems without any *overall dynamic understanding* of our own. Think not? Simply stated, no current model appears to exist which would, for example, provide the *precise* information and the *theoretical basis* for the possible existence of another planet interior to Mercury. Whereas, even in its initial form (*sans* intermediate intervals) this possibility was expressly incorporated in Pierce's initial approach, while in light of present concerns with Global Warming the possible *intermediate* location of Earth becomes more than a mere historical asterisk.

Weakened by special interests, discouraged by behavioural deficiencies and also impeded by disbelief, even the most fundamental question concerning whether Climate Change originates *primarily* from *within*, *i.e*., confines of planet Earth, or from *without* as an integral component of a larger System cannot be tackled adequately at present. Furthermore, what can be made of the location of Earth itself in the intermediate position between Venus and Mars, and what role might this apparent anomaly have played in global warming during the past, distant or otherwise?

All of which is further exacerbated by increasing population growth, unceasing deforestation, rapidly diminishing resources with warfare and mental illness also rising on a Global scale. Truly an Age of Disillusionment and concern.
In the meantime the present inquiry turns next to the initial application of the Pierce Divisor approach to external planetary structures with or without the following suggested guidelines.

PROVISIONAL GUIDELINES FOR EXTERNAL SYSTEMS

Test Format, *Phi*-series Relations and Base Periods

Remaining with the order adopted by Peirce which commences with the outermost PLANET #1 with the greatest period of revolution, moving inwards (by way of Synodic 2-1, then PLANET #2, etc.), will generally involve three consecutive mean periods. All of which can be determined by the following *Phi*-series synodic relations if needed:

Relation (1) The Synodic mean between two bracketing periods of revolution, thus the product of the periods divided by their difference.

Relation (2). Relation (2) requires *two adjacent periods above* to generate *the next value below,* and thereafter generates all further lower periods if or as required.

Relation (4). The geometric mean of any pair of bracketing periods. Thus Relation (4 ± 1), or simply Relation (4) as used.

Relation(4E) Relation (4E ± 2), Relation (4E ± 3), Relation (4E ± 4), Relation (4E ± 5) and Relation (4E ± 6). Such applications depend on specific prior restorations (in due order) above and below the target position(s).

Relation(4F) Relation (4F+3). Special case for PLANET #2 only. Requires both the Base period and periods below #2. This application serves to synchronize the restored periods at this point with the those of the divisor framework

The above relations are provided in Table 4 with the Fibonacci and the Lucas series in vertical and inverted form to match their inclusions in Tables 2a, 2b and also the format adopted for exoplanetary structures.
Lastly, possible departures from the framework are included as variations which may be encountered among external systems. For similar systems, however

(a) Planets may occupy intermediate (synodic) locations, as in the case of Earth.
(b) Planets and adjacent synodic locations may be unfilled (*i.e.*, absent), *e.g.*, the Mars-Jupiter Gap.
(c) Departures from the theoretical framework, or (a) and (b) may indicate disrupted planetary systems.
(d) Planetary systems may possess residual Fibonacci indicators, as in the Solar System.
(e) Planetary systems may also possess residual Lucas indicators for the same reason as (d).

Planets N *Synodics #*	**Divisors** (added)
PLANET 1	**1**
Synodic 2-1	1
PLANET 2	**2**
Synodic 3-2	4
PLANET 3	**6**
Synodic 4-3	9
PLANET 4	**15**
Synodic 5-4	25
PLANET 5	**40**
Synodic 6-5	64
PLANET 6	**104**
Synodic 7-6	169
PLANET 7	**273**
Synodic 8-7	441
PLANET 8	**714**
Synodic 9-8	1156
PLANET 9	**1870**
Synodic 10-9	3025
PLANET 10	**4895**

PHI-SERIES RELATION (1), the Synodic mean: B = AC/| (A - C) |.
For any three successive *Phi*-series periods, A, B, C middle period (B) is the *product of the periods* on either side *divided by their difference*.

PHI-SERIES RELATION (2), the Half-harmonic mean: C = AB/(A + B).
If two upper adjacent periods A, B are known, the third and lower (C) is the *product* of the two adjacent periods *divided by their sum*.

PHI-SERIES RELATION (4), the extended Geometric mean: B = √(AC).
For any three successive periods,the middle period (B) is the *geometric mean* of the periods on either side, as are the resulting periods for the positions ± 2, ± 3,.. = (4E) ; relation (4F+3) pertains to Planet #2 alone.

BASE PERIODS B1/B2.
Base period B1 is the period of the outermost planet as detected.
Base Period B2 may be applicable if *Synodic* 2-1 is marginally > B1.

BASE PERIOD B3.
Approximate base periods (B3s) result from *reversed procedures*, *i.e.* ,the *products* of known periods and their assigned divisors.

BASE PERIOD B4.
Approximate base periods (B4s) can be obtained from the averaged values of the available B3 *products.*

BASE PERIOD B5.
Working, non-standard base periods (B5s) can be applied when none of the above prove to be applicable.

Fibonacci series 1, 1, **2, 3, 5**	
1	
1	4181
2	2584
3	1597
5	610
8	377
13	233
21	144
34	89
55	55
89	34
144	21
233	13
377	8
610	5
987	3
1597	2
2584	1
4181	1
Fibonacci series	

Lucas series 1, **3, 4, 7**	
-	
1	5778
3	3571
4	2207
7	1364
11	843
18	521
29	322
47	199
76	123
123	76
199	47
322	29
521	18
843	11
1364	7
2207	4
3571	3
5778	1
Lucas series	

Table 4. Divisor assignments, numerical series, *Phi*-series relations and conventions for base periods B1 thru B5.

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INTIMATIONS OF COMMONALITY IN PLANETARY SYSTEMS

PART I. The Pierce Planetary Framework (1850) Revisited. (1.16 Mb).
PART II. The Pierce Framework and External Systems. (310 Kb).
PART III. Real-time Motions in the Solar System and the Golden Ratio. (3.19 Mb).
PART IV. The Fibonacci series, the Lucas series and Platonic Triangles. (456 Kb).
PART V. Time and Tide: The Spiral Form in Time and Place. (15.75 Mb).

EXTRA: EXCURSUS (Historical/Mathematical Issues). (2.45 Mb).
OTHER: PART I plus the EXCURSUS. (1.311 Mb).
MAIN: Full text plus the EXCURSUS. (21.44 Mb).

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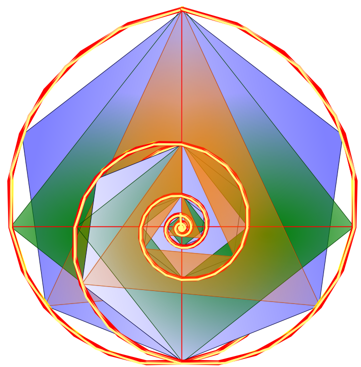
PART TWO



THE PIERCE FRAMEWORK AND EXTERNAL SYSTEMS

EXTERNAL PLANETARY SYSTEMS

Preliminary remarks and initial tests

This research has made use of the Exoplanet Orbit Database and the Exoplanet Data Explorer at exoplanets.org between 2018 and the present (2022).[9] The research was placed on hold when the periods for the first and the last examples originally treated here - GJ 876 and HD 30177 - were changed drastically, which destroyed the associated analyses. These occurrences also caused a reevaluation of the available exo-planet data, resulting in a minor loss of confidence in the same, the coincidental changes notwithstanding. Nevertheless, both analyses have been omitted from the present discussion despite the further insights they provided.

What follows next is a reduced treatment presented on a take-it-or-leave it basis. I have assumed that periods of of "revolution ", "years" and "days" are just that as applied to the exoplanets and have proceeded accordingly. This is necessarily an initial attempt, but at least with standard procedures and a specific planetary framework as its basis.

INITIAL TESTS

2 : 1 and 4 : 2 Octaves

Omitting single planets and deferring multiple configurations until later, dual configurations can include alternate planet-to-planet or synodic-to-synodic pairs, adjacent planet-synodic pairs, and lastly, widely separated pairs of either kind. In particular, the periods and the 2:1 ratios of outermost pairs (Planets 1 and 2) provide base period B1, plus from relation (1) a further base period B2 (*Synodic* 2-1). Since *Synodic* 2-1 is adjacent to Planet 2 theoretical periods below the latter can be derived from the successive applications of relation (2) if required. Either way, a full theoretical framework follows from the division of the selected base period by the standard divisors. A second set of data based on the known periods and *Phi*-series/synodic relations (1), (2) and (4) permits comparison with the latter and the determination of mean and individual errors. This can be useful if uncertainties arise, *e.g.,* selection of outermost planets from either of the 2:1 or 4:2 ratios. The latter – 4:1 in terms of the theoretical framework and the fixed divisors – should not normally be present, but if detected in addition to the 2:1 ratio of the outermost pair, both can be incorporated in a possibly disturbed planetary framework, a primary example being HR 8799.

HR 8799 b-e (b, c and d detected in 2008, e in 2010)[12] Planets 1 and 2, *Synodic* 3-2 and *Synodic* 4-3, Base period B2 = 164,330 days (*Synodic* 2-1). Residual Fibonacci sequence 233-144-89-55-34-21-13-8-5-3-2-1-1 is completed at Planet 14. The restored/suggested overall planetary framework is discussed in detail later.

Period ratios other than 2 : 1

Configurations below the three outermost planets, *i.e*., with period ratios other than 2:1 initially present difficulties, but as a beginning the ratios of detected periods can be compared with those of the Pierce framework divisors and where successful the results can be applied to generate B3 base periods. Before describing the methodology one further element needs to be incorporated in the test procedures. As it turned out, the Fibonacci indicators introduced above became increasingly apparent during the inspection of planetary data, although without the Pierce planetary framework (especially relation (1) that generates the intermediate values) this association might well be dismissed as coincidence. In addition to HR 8799, examples of residual Fibonacci sequences are:

Kepler-460 c-b (2016)[11] Planet #2, adjacent *Synodic* 3-2 respectively, base period B3 = 881.5626 days (2xKepler-460 c). with residual Fibonacci sequence: 55-34-21-13-8-5-3-2-1-1 completed at Planet 8.

Kepler-321 c-b (both planets detected in 2014).[13] Kepler-321 c: 13.093921 days, Kepler-321 b: 4.915379 days. The two Kepler-321 planets detected to date have periods that round to alternate Fibonacci numbers 13 and 5. *Phi*-series/synodic relation (1) provides a synodic difference cycle between Kepler-321c and b of 7.869567 days which rounds in turn to 8 to complete the Fibonacci trio: 5-8-13. Although in years, this sequence is present in Solar System as 5 synodic periods of Venus in 8 years with 13 corresponding periods of revolution for this planet. Also, for Kepler-321, since relation (1) provides an adjacent period to that of Kepler-321b, successive applications of *Phi*-series/synodic relation (2) generate sequential periods below the latter which when rounded complete the Fibonacci sequence 13-8-5-3-2-1-1 at Planet 7. Up to this point the Pierce planetary framework and the associated divisors play no part. All that remains now are the positional assignments for the two planets and determination of the base period. This requires the *ratio* of the detected periods, assignment of the *closest ratio* from the divisor framework, and the selection of a B3 base period from the Divisor-Period products as follows:

The ratio of the Periods of Kepler-321 *c* & Kepler-321 *b* = 2.6638686.
The ratio of the Divisors for Planet 4 & Planet 5 (40/15) = 2.6666667 (Fibonacci 8/3)

The divisors for Planet 4 and Planet 5 are 15 and 40, hence the following B3 base period options with mean errors from Planets 4 through 7 the deciding factor; differences are slight with the average period (B4) a third choice.

15x Kepler-321*c* = 196.408815 days, mean error: - 0.503%.
40x Kepler-321*b* = 196.615160 days, mean error: - 0.608%.

Thus the provisional assignments for Kepler 321*c* and *b* are Planets 4 and 5 with a B3 base period of 15x Kepler 321*c* = 196.40882days. Thereafter the Peirce planetary framework follows from the application of the standard divisors with relations (1) and (2) completing the residual Fibonacci sequence: 13-8-5-3-2-1-1 at planet 7.

PLANETS N *Synodics* #	DIVISOR (added)	PERIODS 1 Base/Divisor	PERIODS 2 B3:Restored	PERIODS 3 B3: Integers	EXOPLANET Kepler-321	%Error (Div:B3)
PLANET 1	1	196.40882	196.40882	196	15x Kepler-321c	Base B3
Synodic 2-1	1	196.40882	202.90383	203	Relation (1)	3.31%
PLANET 2	2	98.204407	99.801751	100	Rel (4F + 3)	1.63%
Synodic 3-2	4	49.102204	50.712538	51	Rel (4E ± 3)	3.28%
PLANET 3	6	32.734803	31.071269	31	Rel (4E ± 4)	-5.08%
Synodic 4-3	9	21.823202	22.630965	23	Relation (1)	3.70%
PLANET 4	15	13.093921	13.093921	13	Kepler-321 c	0.00%
Synodic 5-4	25	7.8563526	7.8695670	8	Relation (1)	0.17%
PLANET 5	40	4.9102204	4.9153790	5	Kepler-321 b	0.11%
Synodic 6-5	64	3.0688877	3.0255822	3	Relation (2)	-1.14%
PLANET 6	104	1.8885463	1.8728064	2	" "	-0.83%
Synodic 7-6	169	1.1621823	1.1567742	1	" "	-0.47%
PLANET 7	273	0.7194462	0.7150872	1	" "	-0.61%
					Mean error:	-0.292%

Table 1. The Divisor framework, Kepler-321 Planets 4 and 5, Base period B3c.

HD 37605 *b-c* (both planets detected in 2014).[4] HD 37605 *c* : 2,720 days. HD 37605 *b* : 55.01307 days.

The suggestion of a Fibonacci presence provided by the near 13-day period of Kepler-321 *c* does not appear to be a coincidence, nor does it appear to be an isolated occurrence. For example, a similar presence is suggested by the lower period of two-planet HD37605, specifically, the approximate 55-day period of HD37650 *b* of 55.01307days. In this system the two detected periods are widely separated and both also occupy *synodic* rather than planetary positions. Nevertheless, though a variation from the successive planetary periods of Kepler-321, relations (1), (2) and (4) are equally applicable to both *Planet* and *Synodic* locations in the *Phi*-series planetary framework. Therefore sequential, multiple applications of these three relations permit the restoration of the seven periods between the two detected planets as shown in Table 7.

ASSIGNMENTS

The ratio of the periods of HD 37605 *c* and HD 37605 *b* = 49.442796.
The ratio of the divisors for *Synodic* 4-3 and *Synodic* 8-7 = 49 (441/9).
The provisional base period (B3) is 9x HD 37605 *c* = 24,480 days.

PLANETS N *Synodics* #	DIVISORS (added)	PERIODS 1 B3/Divisors	PERIODS 2 B3r restoration	PERIODS 3 B3r (rounded)	EXOPLANETS HD 37605	%Error (B1: B3)
PLANET 1	1	24480	24480	24480	9x HD 37605 c	Base B3c
Synodic 2-1	1	24480	25405.81	25406	Relation (1) 12th	3.78%
PLANET 2	2	12240	12467.16	12467	Rel (4F + 3) 11th	1.86%
Synodic 3-2	4	6120	6349.265	6349	Rel (4E ± 3) 10th	3.75%
PLANET 3	6	4080	3899.066	3899	Rel (4E ± 4) 9th	-4.43%
Synodic 4-3	9	2720	2720	2720	HD 37605 c	0.00%
PLANET 4	15	1632	1646.780	1647	Relation (1) 8th	0.91%
Synodic 5-4	25	979.2	1025.754	1026	Rel (4E ± 2) 2nd	4.75%
PLANET 5	40	612	621.0260	621	Relation (1) 7th	1.47%
Synodic 6-5	64	382.5	386.8275	387	Rel (4E ± 4) 1ST	1.13%
PLANET 6	104	235.3846	234.1985	234	Relation (1) 6th	-0.50%
Synodic 7-6	169	144.8521	145.8786	146	Rel (4E ± 2) 3rd	0.71%
PLANET 7	273	89.67033	88.31984	88	Relation (1) 4th	-1.51%
Synodic 8-7	441	55.51020	55.01307	55	HD 37605 b	-0.90%
PLANET 8	714	34.28571	33.89832	34	Relation (2) 5th	-1.13%
Synodic 9-8	1156	21.17647	20.97426	21	" "	-0.95%
PLANET 9	1870	13.09091	12.95715	13	" "	-1.02%
Synodic 10-9	3025	8.092562	8.009294	8	" "	-1.03%
PLANET 10	4895	5.001021	4.949701	5	" "	-1.03%
Synodic 11-10	7921	3.090519	3.059158	3	" "	-1.01%
PLANET 11	12816	1.910112	1.890646	2	" "	-1.02%
Synodic 12-11	20736	1.180556	1.168488	1	" "	-1.02%
PLANET 12	33552	0.729614	0.722164	1	" "	-1.02%
					Mean value	0.212%

Table 2.The Divisor planetary framework and HD 37605, *Synodic* 4-3 & *Synodic* 8-7, Base B3c

The two widely separated planets HD 37605 *c* and *b* with a period ratio of 49.442796 (2720 days/55.01307 days) are readily equated with the Peirce planetary framework and the ratio of the divisors for *Synodic* 4-3 and *Synodic* 8-7 (441 and 9 respectively). The lowest mean error is obtained from product of the period of HD 37605 c and the the divisor for *Synodic* 4-3 (9) = 24,480 days. The odd number of periods between the two detected planets permits the theoretical restoration of all periods between HD37605 *c* and *b* plus all those below the latter. This is feasible since relation (4) – the extended geometric mean – can be applied three times, first at the midpoint between the two planets (at *Synodic* 6-5) then twice more between two new mid-points to obtain the periods for *Synodic* 5-4 and *Synodic* 7-6. This fills three of the seven positions with those remaining determined by relation (1), including the period adjacent to HD 37605 b, thus permitting the generation of the periods for Planets 4, 5 and 6, and finally the completion of the residual Fibonacci sequence: 55-34-21-13-8-5-3-2-1-1 at Planet 12 by use of Relation (2). Another example involving both a Fibonacci indicator (5 versus 5.41608 days) and multiple applications of relation (4) and (2) is given below *sans* table.

HATS-59 *b-c* [5] HATS-59 *c*: 1422 days, HATS-59 *b*: 5.41608 days.
Here the more widely separated periods of HATS-59 *c* and *b* (1422 and 5.41608 days respectively) can be assigned to Planet 1 and Planet 7 with a separation of eleven intervening periods. Again, the odd number permits multiple applications of relation (4) to determine the period of mid-point Planet 4 followed by mid-point periods on either side belonging to *Synodic* 3-2 and *Synodic* 6-5. In this instance there are two positions between the determined periods, not one, therefore relation (1) is not applicable. Instead, with the Pierce planetary framework available, the period of Planet 2 of 711 days can be introduced above *Synodic* 3-2 to allow relation (2) to end at Planet 9.

ASSIGNMENTS

The ratio of the periods of HATS-59 *c* and HATS-59 *b* = 262.551465.
The ratio of the divisors for Planet 1 & Planet 7 (273/1) = 273 (3.83%).
The provisional base period is B1, Planet 1, HATS-59 *c* (1422 days).
The residual Fibonacci presence 13-8-5-3-2-1-1 including HATS-59 *b* is completed at Planet 9.

Variations and Additions

Applying the above procedures to other systems brought to light additional numerical sequences including the double-Fibonacci series,. *i.e.,* instead of Fibonacci 13-8-5, the sequence 26-16-10, etc. The second occurrence, the replacement of the residual Fibonacci series by the Lucas series (1-3-4-7-11-18-29-47-76-123,... etc.,) featured one common departure, namely the inclusion of the number 2 below the sequence 7-4-3. Examples of residual Lucas series present among the available external systems are:

Kapteyn's *c-b* (2014)[6] Kapteyn's *c:* 124.54 days, Kapteyn's *b*: 48.616 days.
No series is initially indicated with the Lucas sequence only becoming apparent after the assignment of the base period and the derivation of the divisor framework. The two periods are again consecutive synodic locations, *i.e*., *Synodic* 5-4 and *Synodic* 6-5 respectively.

ASSIGNMENTS

The ratio of the divisors for *Synodic* 5-4 and *Synodic* 6-5 = 2.56. (square of 1.6, or Fibonacci 5/3)
The ratio of the periods of Kapteyn's *c* and Kapteyn's *b* = 2.5617081.
The provisional base period is B3 = 25x Kapteyn's c = 3113.5 days.
The residual Lucas series 18-11-7-4-3- {2}-1-1 is complete at Planet 10 with {2} anomalous.

Other external systems featuring Lucas sequences include the following, all with {2} anomalous:

Nu Ophiuci *c-b* (2010)[7] Planets 1 and 3 respectively, Base period B1 = 3,186 days (as detected).
Residual Lucas sequence: 18-11-7-4-3- {2}-1-1 completed at Planet 10.
XO-2s *b-c* (2014)[8] Planets 3 and 5 respectively, Base Period B3 = 724.8 days (6x XO-2s c).
Residual Lucas sequence: 18-11-7-4-3-{2}-1-1 completed at *Synodic* 9-8.
Kepler-49 *c-b* (2012)[9] *Synodic* 3-2 and Planet 3 respectively, Base period B3 = 43.6517372 days (4x Kepler-49 c).
Residual Lucas sequence: 11-7-4-3- {2}-1-1 completed at *Synodic* 8-7
Kepler-198 *c-b* (2014)[10] Planets 3 and 4 respectively, Base period B3 = 273.404496 days (6x Kepler-198 c).
Residual Lucas sequence: 29-18-11-7-4-3- {2}-1-1 completed at Synodic 8-7.
Kepler-396 *c-b* (2014)[11] Planet 2 and *Synodic* 3-2 respectively, Base period B3 = 177.01 days (2x Kepler-396 c).
Residual Lucas sequence: 29-{17}-11-7-4-3- {2}-1-1 completed at Planet 7 with {17} for Lucas number 18.
HD 60532 *b-c* (2008)[12] Planets 2 and 3 respectively, Base period B3 = 1214.12 days (2x HD 60532 c).
Residual Lucas sequence: 76-47-29-18-11-7-4-3-{2}-1-1 completed at Planet 9.
HD 163607 *b-c* (2011)[13] Planets 3 and 6 respectively, Base Period B3 = 7,884 days (6x HD 163607 c).
Residual Lucas sequence: 76-47-29-18-11-7-4-3-{2}-1-1 completed at Planet 11.

Finally, partial residual sequences, *i.e*., confined to three consecutive values occur in some instances, while other systems, *e.g.,* TRAPPIST-1 treated next have no immediately discernable sequence.

TRAPPIST-1 (*b* - *g* detected in 2016,[14] TRAPPIST-1 *h* detected in 2017).[15]
TRAPPIST-1*h* : 18.767 days[26]
TRAPPIST-1*g* : 12.35294 days
TRAPPIST-1*f* : 9.206690 days
TRAPPIST-1*e* : 6.099615 days
TRAPPIST-1*d* : 4.049610 days
TRAPPIST-1*c* : 2.4218233 days
TRAPPIST-1*b* : 1.51087081 days
Base period B3, Planet #1: 36.82876 days (4x TRAPPIST-1f)

Initially the period of TRAPPIST-1h was thought to range between14 and 35 days with the absence of a precise base period preventing generation of a divisor-based planetary framework. Nevertheless the latter still appears to be present in the structure of TRAPPIST-1 as indicated by the sequential twinned reduction ratios for both two-third ratios and also the upper three-fifth ratio. Thus the Pierce ratios have a key role to play in the present example. In more detail, commencing with the 9.206690-day period of TRAPPIST-1*f* the successive reduction ratios generate ordered approximations for the periods of the remaining four TRAPPIST-1planets1*e* through1*b*. Furthermore, the applicationof *Phi*-series relation(4) to the periods of TRAPPIST-1*f* and TRAPPIST-1*d* results in a period of 6.081193 days versus the 6.0992672-period of TRAPPIST-I *e*.

Although the twinned Pierce ratios appear to be reflected in a substantial part of the structure of TRAPPIST-1 it appears that with respect to the theoretical framework the five adjacent planets TRAPPIST-1*f* through 1b occupy consecutive planetary/synodic positions. Thus while the Solar System has one planet (Earth) in a synodic location between adjacent planets, TRAPPIST-1 appears to have at least three in ordered succession, two of which can be approximated (if not confirmed) by relation(1). In short, the general synodic formula can be applied to the mean periods of revolution of TRAPPIST-1*c* and 1*e* to approximate the difference period in the current location of 1*d*, and and similarly applied to TRAPPIST 1*b* and 1*d* for the difference period in the current location of 1*c*.

PLANETS Synodics	N #	RATIOS (Pierce)	DIVISOR (added)	DIVISOR RATIO Results/Ratios	B1-RATIOS B1s/Divisors	PERIODS 1 B3/Divisors	PERIODS 2 B1s(actual)	EXOPLANETS TRAPPIST-1	%ERR (B1: B3)
PLANET	1	1/1	1	(1/1)	.	36.826760	.	4x TRAPPIST-1f	Base 3
Synodic	?	1/1	2	2 (2/1)	1.9623147	18.413380	18.767	TRAPPIST-1h	1.92%
PLANET	?	1/2	3	3 (3/1)	2.8912142	12.275587	12.35294	TRAPPIST-1g	0.63%
Synodic	?	1/2	4	4 (4/1)	4	9.20669	9.20669	TRAPPIST-1f	0.00%
PLANET	3	2/3	6	3 (6/2)	6.0378991	6.1377933	6.0992672	TRAPPIST-1e	-0.63%
Synodic	4-3	2/3	9	2.25 (9/4)	9.1683135	4.0918622	4.0167431	TRAPPIST-1d	-1.84%
PLANET	4	3/5	15	2.5 (15/6)	15.206213	2.4551173	2.4218233	TRAPPIST-1c	-1.36%
Synodic	5-4	3/5	25	2.777* (25/9)	24.374526	1.4730704	1.5108708	TRAPPIST-1b	2.57%
PLANET	5	5/8	40	2.666*(40/15)	39.580739	0.9206690	0.9304212	(Rel.2)	1.06%
Synodic	6-5	5/8	64	2.56 (64/25)	63.955265	0.5754181	0.5758206	“ “	0.07%
PLANET	6	8/13	104	2.6 (104/40)	103.53600	0.3541035	0.3556904	“ “	0.45%
Synodic	7-6	8/13	169	2.6406 (169/64)	167.49127	0.2179098	0.2198727	“ “	0.90%
PLANET	7	13/21	273	2.625 (273/104	271.02727	0.1348966	0.1358784	“ “	0.73%
Synodic	8-7	13/21	441	2.609 (441/169)	438.51854	0.0835074	0.0839799	“ “	0.57%
PLANET	8	21/34	714	2.615 (714/273)	709.54581	0.0515781	0.0519019	“ “	0.63%
								Mean error:	0.407%

Table 3. The Pierce planetary framework, TRAPPIST-1, compressed adjacent planets, Base period B3

Irrespective of additional complications that arise from the dual occurrence of TRAPPIST-1*d* and 1*c* in both synodic computations the first two-thirds reduction ratio nonetheless serves to synchronize TRAPPIST-1*e* with divisor planet #3 and therefore all the remaining positions. At which point the simplest option for a theoretical base period is to reverse standard procedures and use the products of the known periods and their associated divisors to generate B3 estimates, with TRAPPIST-1*f* (4x = 36.82676 days) the closest to the mean value.

More recently,[16] a period of 18.767 days has been deduced for TRAPPIST-1*h* with a resulting *Synodic* 2-1 interval of 36.14366 days between the latter and the 12.35294-day period of TRAPPIST-1*g*. The last period could be applied as a provisional base period B2. Lastly, though the12.35294-day period of TRAPPIST-1*g* appears to be anomalous, the synodic period between this value and any of the 36-day estimates for base period B3 results in values between 18.356 and 18.767 days. All of which raises the possibility that if disruptions of TRAPPIST-1 may have occurred, that they might have involved the two outermost planets. If so there would be no perceptible gap *per se*, but absence of the outermost planet (or both) and possible readjustments by the others. Whether this scenario would be drastic enough to cause TRAPPIST-1*b* thru 1*f* to occupy adjacent sidereal/synodic locations *en masse* or cause the anomalous period of 1*g* is another matter. Then again, this scenario may also be a more natural occurrence with compression a component of later phases in the life-cycle of this particular System itself.

Further candidates for compressed systems

Additional systems with apparent planet/synodic compression and residual Fibonacci/Lucas sequences include:
YZ Cet *d-c-b* (2017) [17] Synodic 3-2, Planet 3, *Synodic* 4-3. Base B3 = 18.62508 days
(4x YZ Cet *d*). Residual Fibonacci sequence: 5-3-2-1-1 completed at *Synodic* 5-4.
Kepler-23 *d-c-b* (*b-c* 2012, *d* 2014) [18],*Synodic* 3-2, Planet 3, *Synodic* 4-3, Planet 4. Base B3 = 61.097216 days
(4xKepler-23 *d*). Residual Lucas sequence: 11-7-4-3- {2}-1-1 completed at Planet 6.
Kepler-37 *d-c-b* (2016)[19] *Synodic* 4-3, Planet 4, *Synodic* 5-4. Base B3 = 358.129683 days.
(9x Kepler-37 *d*). Residual Fibonacci sequence: 21-13-8-5-3-2-1-1 completed at *Synodic* 8-7.
Kepler-107 *e-d-c-b* (2014)[20] Planet 2, *Synodic* 3-2, Planet 3, *Synodic* 4-3. Base B3 = 29.498352 days
(2x Kepler-107 *e*). Residual Fibonacci sequence: 8-5-3-2-1-1 completed at Planet 5.
Kepler-184 *d-c-b* (2014)[21] *Synodic* 5-4, Planet 5, *Synodic* 6-5. Base B3 = 203.304324 days
(25x Kepler-184 *d*). Residual Lucas sequence: 11-7-4-3- {2}-1-1 completed at *Synodic* 9-8.

Kepler-208 *e-d-c-b* (2014)[22] *Synodic* 3-2, Planet 3, *Synodic* 4-3, Planet 4. Base B3 = 65.0783 days (4x Kepler-208 *d*). Residual Lucas sequence: 11-7-4-3- {2}-1-1 completed at Planet 6.
Kepler-295 *d-c-b* (2014) [23] Planet 3, *Synodic* 4-3, Planet 4. Base B3 = 203.304324 days (6x Kepler-295 *d*). Residual Fibonacci sequence: 34-{22}-13-8-5-3-2-1-1 completed at Planet 7.
Kepler-374 *d-c-b* (2014)[24] Planet 3, *Synodic* 4-3, Planet 4. Base B3 = 30.169314 days (6x Kepler-374 *d*). Residual Fibonacci sequence: 5-3-2-1-1 completed at Planet 5.
Kepler-446 *d-c-b* (2014)[25] *Synodic* 4-3, Planet 4, *Synodic* 5-4. Base B3 = 46.340289 days (9x Kepler-446 *d*). Residual Fibonacci sequence: 5-3-2-1-1 completed at Synodic 6-5.
Kepler-758 *e-d-c-b* (2016)[26], Planet 4, *Synodic* 5-4, Planet 5, *Synodic* 6-5. Base B3 = 307.4493 days (15xKepler-758 *d*). Residual Fibonacci sequence: 8-5-3-2-1-1 completed at *Synodic* 8-7.

HR 8799 (HR 8799 *b, c* and *d* detected in 2008, HR 8799 *e* detected in 2010.[2]
Initially detected as a three-planet system, HR 8799 has known [3] 1:2 and 1:4 resonances already subject to detailed analyses.[27, 28] The theoretical planetary framework is augmented by a possible 1:9 resonance following the discovery in 2010 of a fourth planet (HR 8799 *e*) with a period of 18,000 days[38] versus 18,250 days for the associated resonance.

ASSIGNMENTS (MEAN PERIODS: Days/Julian Years)
HR 8799 *b* : 164,250 days/449.691991786 years. HR 8799 *c* : 82,145 days/224.900752909 years.
HR 8799 *d* : 41,054 days/112.399726215 years. HR 8799 *e* : 18,000 days/49.2813141684 years.

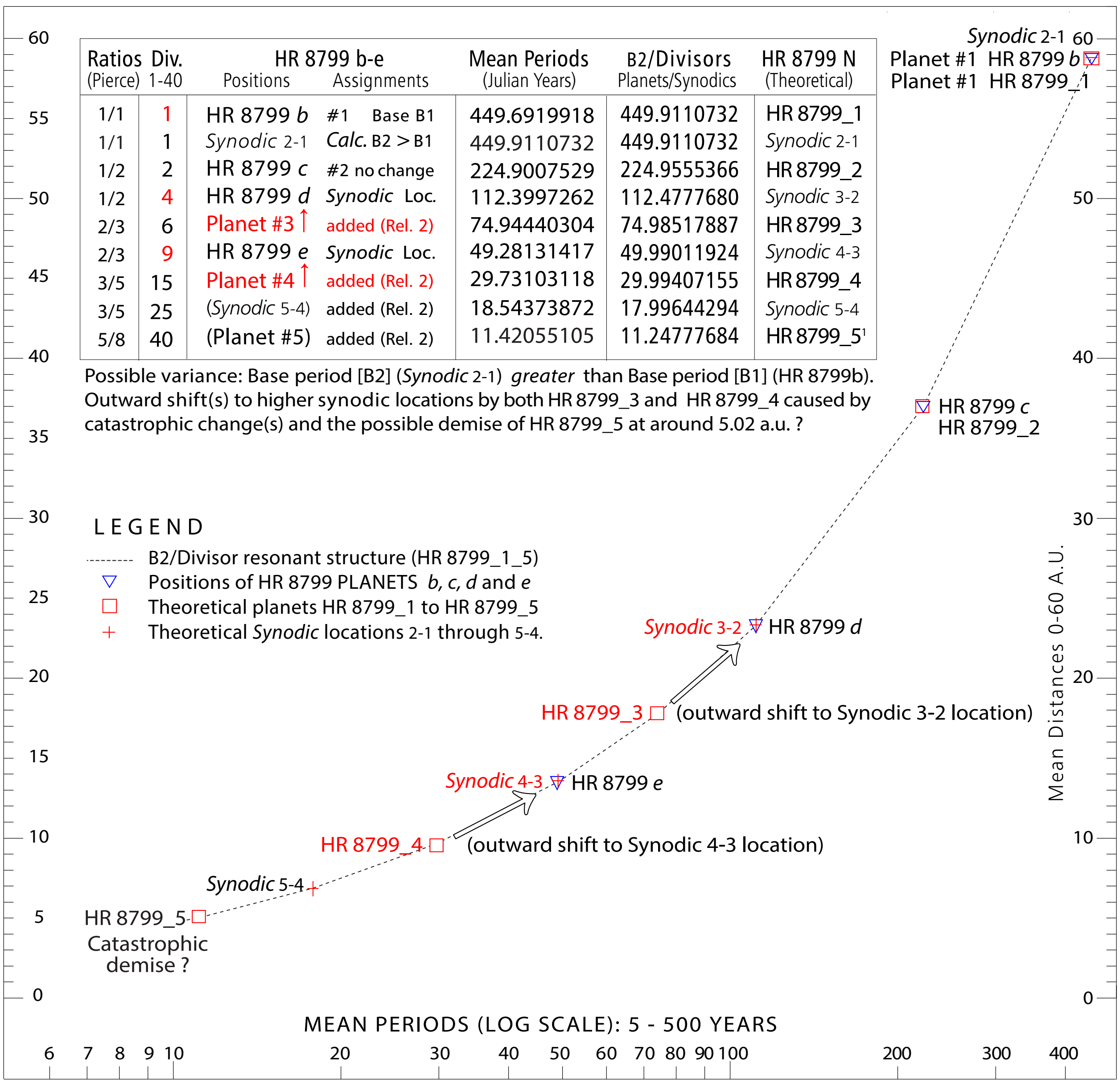
The ratio of the Divisors for Planet 1 and Planet 2 (2/1) = 2.00000; second octave (HR 8799 *c* and *d*) is also present. Ratio of the periods of HR 8799 *b* and c is 1.999613; the 4/2 ratio of the second octave HR 8799 *c* and *d* = 2.000901. The provisional base period (B2) is *Synodic* 2-1(164330.019 days) between HR 8799 *b* and *c* (lowest mean error).

PLANETS N *Synodics* #	RATIOS (Pierce)	DIVISORS (added)	PERIODS 1 B2/Divisors	PERIODS 2 B1 Actual	PERIODS 3 B1/(Rounded	EXOPLANETS HR 8799	%ERROR (B1: B2)
PLANET 1	1/1	1	164330.019	164250	164250	HR 8799 *b*	-0.049%
Synodic 2-1	1/1	1	164330.019	164330.019	164330	(Rel. 1)	0.000%
PLANET 2	1/2	2	82165.0097	82145	82145	HR 8799 *c*	-0.024%
Synodic 3-2	1/2	4	41082.5049	41054	41054	HR 8799 *d*	-0.069%
PLANET 4	2/3	6	27388.3366	27373.443	27373	(Rel. 2)	-0.054%
Synodic 4-3	2/3	9	18258.8911	18000	18000	HR 8799 *e*	-1.418%
PLANET 4	3/5	15	10955.3346	10859.2591	10859	(Rel. 2)	-0.877%
Synodic 5-4	3/5	25	6573.20078	6733.10057	6773	“ “	3.041%
PLANET 5	5/8	40	4108.25049	4171.35627	4171	“ “	1.536%
Synodic 6-5	5/8	64	2567.65655	2581.49088	2581	“ “	0.539%
PLANET 6	8/13	104	1580.09634	1594.63378	1595	“ “	0.920%
Earth/*Syn* 7-6	8/13	169	972.366979	985.730287	986	“ “	1.374%
PLANET 7	13/21	273	601.941463	609.169393	609	“ “	1.201%
Synodic 8-7	13/21	441	372.630430	376.498114	376	“ “	1.038%
PLANET 8	21/34	714	230.154089	232.686099	233	“ “	1.100%
Synodic 9-8	21/34	1,156	142.153996	143.808516	144	“ “	1.164%
PLANET 9	34/55	1,870	87.8770158	88.8784095	88	“ “	1.140%
Synodic 10-9	34/55	3,025	54.3239734	54.9299112	55	“ “	1.115%
PLANET 10	55/89	4,895	33.5709948	33.9485443	34	“ “	1.125%
Synodic 11-10	55/89	7,921	20.7461204	20.9813561	21	“ “	1.134%
PLANET 11	89/144	12,816	12.8222550	12.9671908	13	“ “	1.130%
Synodic 12-11	89/144	20,736	7.92486591	8.01416473	8	“ “	1.127%
PLANET 12	144/233	33,552	4.89777121	4.95302617	5	“ “	1.128%
Synodic 13-12	144/233	54,289	3.02694873	3.06113853	3	“ “	1.130%
PLANET 13	233/377	87,841	1.87076672	1.89188765	2	“ “	1.129%
Synodic 14-13	233/377	142,129	1.15620330	1.16925087	1	“ “	1.128%
PLANET 14	377/610	229,970	0.71457155	0.72263678	1	“ “	1.129%
						Mean error:	0.809%

Table 4. The Pierce planetary framework and HR 8799; Planets 1 and 2, Synodics 3-2 and 4-3. Bases B1 and B2.

HR8799 AS A DISTURBED PLANETARY SYSTEM

As in the case of the Solar System, base period B2 (164330.019 days, *Synodic* 2-1 between HR 8799 *c* and HR 8799 *b*) is marginally greater than base period B1 (164,250 days). This, allied with 1:4 and 1:9 resonances recognizable as squares belonging to two successive synodic positions in the divisor framework suggests that HR8799 may also be a disrupted system as seen in Table 4 and Figure 1 below. If so, it is perhaps possible that HR 8799 *d* and HR 8799 *e* may currently be occupying *Synodic* 4-3 and *Synodic* 3-2 locations resulting from outward orbital shifts of divisor planets #3 and #4. In which case theoretical mean periods of revolution for the latter pair can be approximated by either successive applications of relation (2) to HR 8799 *c* and HR 8799 *d,* and (or) the application of divisors 1 to 40 to base period B2. Whether a theoretical planet at or near position #5 (~ 4.5 HR 8799 standard mass?) suffered a catastrophic demise is hypothetical, but still a possibility which can be considered further in terms of the debris field in the inner region of HR 8799 discussed by Moore and Quillen (2013),[29] Contro *et al*, (2014)[30] and Contro *et al,* (2016) [31]. In particular, the theoretical distance for possibly defunct HR_8799_5 at ~5.02 a.u. is situated reasonably close to the " inner and outer edges, located at ~6 and ~8 au," of the debris belt discussed by the latter authors.

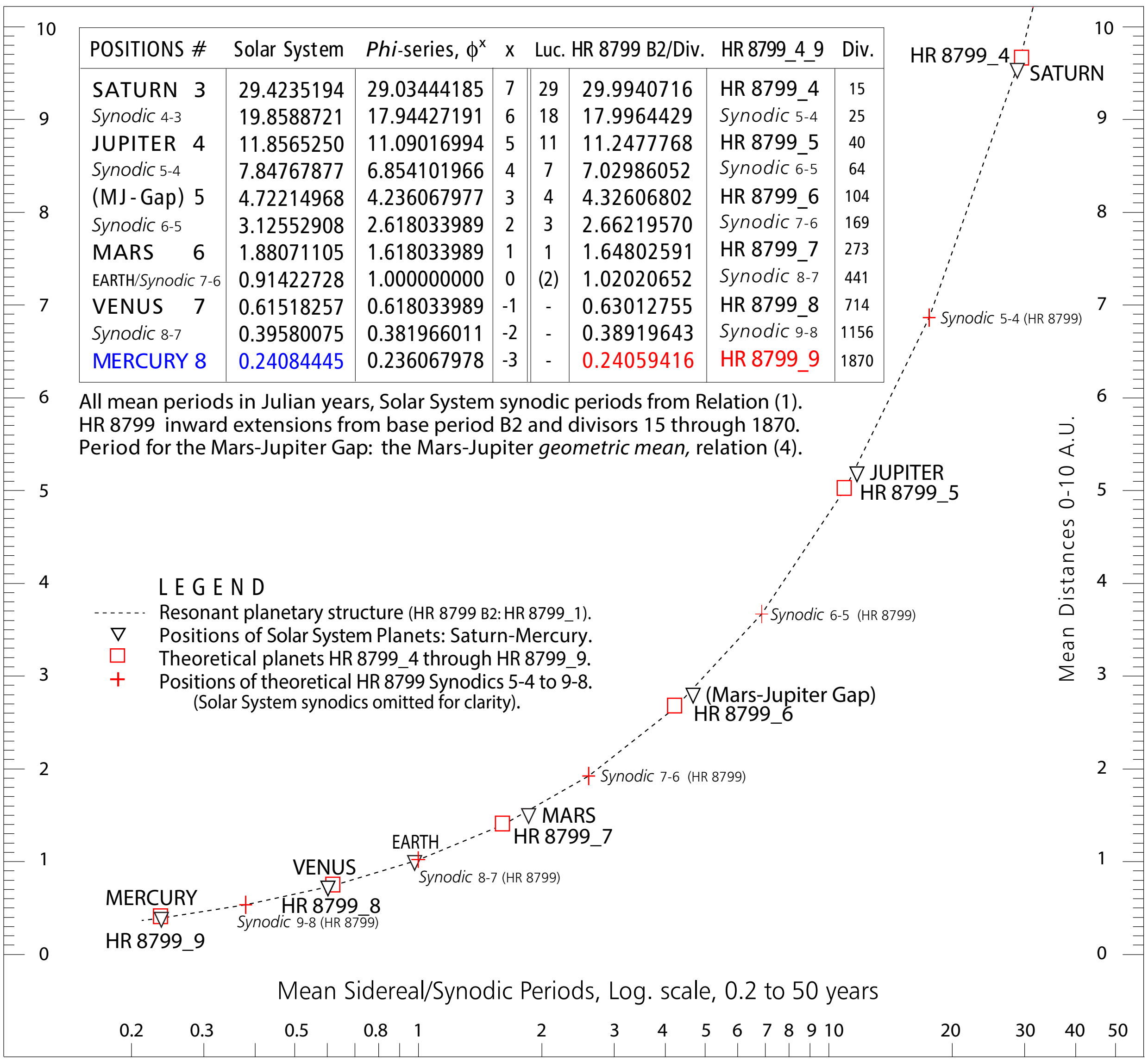


Ratios (Pierce)	Div. 1-40	HR 8799 b-e Positions	Assignments	Mean Periods (Julian Years)	B2/Divisors Planets/Synodics	HR 8799 N (Theoretical)
1/1	1	HR 8799 *b*	#1 Base B1	449.6919918	449.9110732	HR8799_1
1/1	1	*Synodic* 2-1	*Calc.* B2 > B1	449.9110732	449.9110732	*Synodic* 2-1
1/2	2	HR 8799 *c*	#2 no change	224.9007529	224.9555366	HR8799_2
1/2	4	HR 8799 *d*	*Synodic* Loc.	112.3997262	112.4777680	*Synodic* 3-2
2/3	6	Planet #3 ↑	added (Rel. 2)	74.94440304	74.98517887	HR8799_3
2/3	9	HR 8799 *e*	*Synodic* Loc.	49.28131417	49.99011924	*Synodic* 4-3
3/5	15	Planet #4 ↑	added (Rel. 2)	29.73103118	29.99407155	HR8799_4
3/5	25	(*Synodic* 5-4)	added (Rel. 2)	18.54373872	17.99644294	*Synodic* 5-4
5/8	40	(Planet #5)	added (Rel. 2)	11.42055105	11.24777684	HR8799_5[1]

Fig 1. HR 8799 *b* to *e* and departures from the Pierce planetary framework for planets 1 through 5.

In any event, if this scenario is valid there is a distinct similarity between HR 8799 and the Solar System, since–for whatever reasons–the fifth planet in both systems (counting inwards) can be considered to be absent. Also, in addition to the known planetary resonances, similarities between the two systems, especially with respect to the gas giants Jupiter and Saturn have already been noted by Fabrycky and Murray-Clay (2010).[31] In fact, the similarity between the two may be greater than already suspected, as shown in log-linear Figure 2 with the Solar System in an eight-planet configuration and HR 8799_1_9 as a theoretical nine-planet system. Or, as a substitute, depending on what might have taken place and the original mass of planet HR 8799_5, major compensatory adjustments that may have occurred in another eight-planet system.

The linkage between the divisors from the Fibonacci-based Peirce approach, the *Phi*-series planetary framework and the Lucas series – all with respect to unity and the mean parameters of Earth – is also shown in Figure 2. Here the eight-planet Solar System from Saturn to Mercury is compared to a theoretical inward extension of HR 8799 (HR 8799_4 thru HR 8799_9) with the latter represented as a nine-planet system.



POSITIONS #	Solar System	*Phi*-series, ϕ^x	x	Luc.	HR 8799 B2/Div.	HR 8799_4_9	Div.
SATURN 3	29.4235194	29.03444185	7	29	29.9940716	HR 8799_4	15
Synodic 4-3	19.8588721	17.94427191	6	18	17.9964429	*Synodic* 5-4	25
JUPITER 4	11.8565250	11.09016994	5	11	11.2477768	HR 8799_5	40
Synodic 5-4	7.84767877	6.854101966	4	7	7.02986052	*Synodic* 6-5	64
(MJ-Gap) 5	4.72214968	4.236067977	3	4	4.32606802	HR 8799_6	104
Synodic 6-5	3.12552908	2.618033989	2	3	2.66219570	*Synodic* 7-6	169
MARS 6	1.88071105	1.618033989	1	1	1.64802591	HR 8799_7	273
EARTH/*Synodic* 7-6	0.91422728	1.000000000	0	(2)	1.02020652	*Synodic* 8-7	441
VENUS 7	0.61518257	0.618033989	-1	-	0.63012755	HR 8799_8	714
Synodic 8-7	0.39580075	0.381966011	-2	-	0.38919643	*Synodic* 9-8	1156
MERCURY 8	0.24084445	0.236067978	-3	-	0.24059416	HR 8799_9	1870

Fig. 2. HR 8799_4 to HR8799_9, extended Pierce resonances and the Solar System from Saturn to Mercury.

In the above configuration Solar System base B2 is 171.4442890 years, HR 8799 base B2 is 449.9110736 years and HR 8799 base B1 is 449.69199818 years. The similarity between the two Systems, the Lucas Series and lower *Phi*-series gives rise to the following relations involving the limiting Pierce reduction ratio ϕ^{-2} (0.38196601125) and reciprocal ϕ^2 (2.61803398875), the outward *Pheidian* constant for the periods of revolution, thus:

Solar System, B2	=	171.4442890 years	$\phi^2 \cdot$ B2, Solar System	=	448.8469772 years
Base B1, HR 8799$\cdot\phi^{-2}$	=	171.8507380 years	Base B1, HR 8799 b	=	449.6919918 years
Base B2, HR 8799$\cdot\phi^{-2}$	=	171.7670564 years	Base B2, HR 8799	=	449.9110736 years

and the following comparison:

PLANETS N *Synodics* #	DIVISORS 1- 714	Sol. System II B2/Div. 1-714	HR 8799 B2/ϕ^2	LUCAS N	Phi-Series ϕ^x (Periods/years)	Exp. x
NEPTUNE 1	1	171.444290	171.767283		199.0050249	11
Synodic 2-1	1	171.444290	171.767283		122.9918694	10
URANUS 2	2	85.7221448	85.8836413		76.01315562	9
Synodic 3-2	4	42.8610724	42.9418207		46.97871376	8
SATURN 3	6	28.5740483	28.6278804		29.03444185	7
Synodic 4-3	9	19.0493655	19.0852536		17.94427191	6
JUPITER 4	15	11.4696193	11.4511522	11	11.09016994	5
Synodic 5-4	25	6.85777158	6.87069131	7	6.854101966	4
(MJ-Gap) 5	40	4.28610724	4.29418207	4	4.236067977	3
Synodic 6-5	64	2.67881702	2.68386379	3	2.618033988	2
MARS 6	104	1.64850278	1.65160849	(2)	1.618033988	1
Earth/*Syn* 7-6	169	1.01446325	1.01637445	1	1.000000000	0
VENUS 7	273	0.62800106	0.62918419	1	0.618033988	-1
Synodic 8-7	441	0.38876256	0.38949497	-	0.381966011	-2
MERCURY 8	714	0.24011805	0.24057042	-	0.236067977	-3

Table 5. Solar System and HR 8799 with emphasis on the *Lucas* & *Phi*-Series.

All of which is encouraging enough to precipitate a return to the Solar System and an investigation of these two fundamental constants and related variants with respect to real-time varying motions of the planets, commencing with those of Jupiter and Saturn followed by the terrestrial planets and the remaining two superior planets.

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