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PROJECTILES, PARABOLAS, AND VELOCITY EXPANSIONS OF THE LAWS OF PLANETARY MOTION

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ABSTRACT

Kepler's Third Law of planetary motion: $T^2 = R^3$ (T = period in years, R = mean distance in astronomical units) may be extended to include the inverse of the mean speed V_i (in units of the inverse of the Earth's mean orbital speed) such that:

$$R = V_i^2$$

and

$$T^2 = R^3 = V_i^6$$

The first relationship - found in Galileo's last major work, the *Dialogues Concerning Two New Sciences* (1638), - may also be restated and expanded to include relative speed V_r (in units of Earth's mean orbital speed k) and absolute speed $V_a = kV_r$, thus:

$$T = V_i^3$$

$$V_i = T/R$$

$$V_r = R/T$$

$$V_a = kR/T$$

$$V_r = kR^{-1/2}$$

$$V_r = kT^{-1/3}$$

$$V_a = kR^{-1/2}$$

$$V_a = kT^{-1/3}$$

This paper explains the context of Galileo's velocity expansions of the laws of planetary motion and applies these relationships to the parameters of the Solar System. A related "percussive origins" theory of planetary formation is also discussed.

1. Introduction. Galileo's velocity expansions of the Third Law of planetary motion are buried in his *Dialogues Concerning Two New Sciences* in two adjoining passages concerned with parabolas and the trajectories of projectiles.[1] In the first passage Galileo defines a semi-parabola in terrestrial contexts, but in the second he states in an unexpected historical aside that this semi-parabola pertains to the *origins* and *velocities* of the *planets*. In addition, Galileo states here that he has successfully tested the second application on the Solar System, and further, that he has provided all the materials necessary for the reader to verify the latter application.

The semi-parabola proves to be applicable to planetary motion as Galileo claimed, while the integral velocity variants of the laws of planetary motion and the implications of Galileo's application lead in turn to an examination of Galileo's "percussive origins" theory of planetary formation.

2. The Parabola. The parabolas used by Galileo initially describe the paths followed by projectiles in terrestrial applications. In this context Galileo elects to standardize his procedures on the grounds that an infinite number of uniform horizontal velocities may be compounded with the " naturally accelerated" velocity of a falling body. Accordingly, Galileo combines accelerated velocity on the vertical axis with a specific uniform velocity on the horizontal axis to create a semi-parabola with the vertex at the origin and a distance of "four" units between the vertex and the directrix.[2]

This semi-parabola apparently has a second function, however, for following its {p. 209} construction Galileo states in the dialogue[3] that he is returning to the subject at hand only to embark on a historical aside dealing with:[4]

> " ...the beautiful agreement between this thought of the Author (Galileo) and the views of Plato concerning the origin of the various uniform speeds with which the heavenly bodies revolve" (italics supplied).

The relationship between the parabola, the "views of Plato," and planetary velocity is described in detail in the ensuing dialogue. At the conclusion Galileo states that he has applied the parabola to planetary motion and that:

> *he once made the computation and found a satisfactory correspondence with the observation*. But he did not wish to speak of it, lest in view of the odium which his many new discoveries had already brought upon him, this might be adding fuel to the fire. *But if anyone desires such information he can obtain it for himself from the theory set forth in the present treatment*.[4] (italics supplied)

From the last part of this passage it thus appears that Galileo successfully tested the new application on the parameters of the Solar System. Moreover, Galileo also asserts here that he as provided sufficient information for the reader to verify his results.

Once alerted to Galileo's intention, it becomes clear that the inclusion of the " views of Plato " on the subject is a relevant and necessary device permitting the use of parameters and concepts common to both applications.[5] Somewhat surprisingly, the semi-parabola is directly applicable to the Solar System with no modification at all, once the heliocentric concept is invoked and the frames of reference are understood to be provided by the mean period, the mean distance, and the mean velocity of *Earth*, i.e., *unity* in all cases. With these two provisos Galileo's semi-parabola may then be treated in terms of planetary motion with the *mean distance R* from the SUN represented by the "distance" down the vertical axis, and the inverse (V_i) of the mean planetary speed V_r represented by the "uniform velocity" along the horizontal axis. Galileo's semi-parabola thus demonstrates the relationship:

$$R = V_i^2 \tag{1}$$

while the further relationships:

$$V_r = R/T \tag{2}$$

$$V_i = T/R \tag{3}$$

$$T = V_i^3 \tag{4}$$

$$V_i^6 = R^3 = T^2 \tag{5}$$

where T is the sidereal period in years, follow from Kepler's Third Law of planetary motion. Relation (5) may also be expressed in exponents (i.e., $[V_i^0, V_i^1, V_i^2, V_i^3]$) {p. 210} and applied to the parabola as *the first three integer sets which illustrate the Third Law*: [7]

$V_i^0\ V_i^1\ V_i^2\ V_i^3$	(V_i, R)	[*1*,(V_i, *R*),*T*]	V_i^6 =	R(A.U.)^3 =	T (years)^2 =	N
[1, 1, 1, 1]	(1, 1)	1,(1, 1), 1	1^6 =	1(A.U.)^3 =	1 (year)^2	Earth
[1, 2, 4, 8]	(2, 4)	1,(2, 4), 8	2^6 =	4(A.U.)^3 =	8 (years)^2 =	64
[1, 3, 9, 27]	(3, 9)	1,(3, 9), 27	3^6 =	9(A.U.)^3 =	27 (years)^2 =	729

The parabola in the standard context is illustrated in figure 1a, and in the astronomical context with (V_i, R) as the subset of [*1*, (V_i, *R*), *T*] in figure 1b.

For absolute velocity (V_a), Relation (2) may be modified such that:

$$V_a = kR/T \tag{2a}$$

where k is the mean velocity of Earth. With $k = 29.79$ kilometers per second for this constant we obtain the comparison with modern estimates (Table 1). The superior planets are shown on the inverted parabola in figure 2.

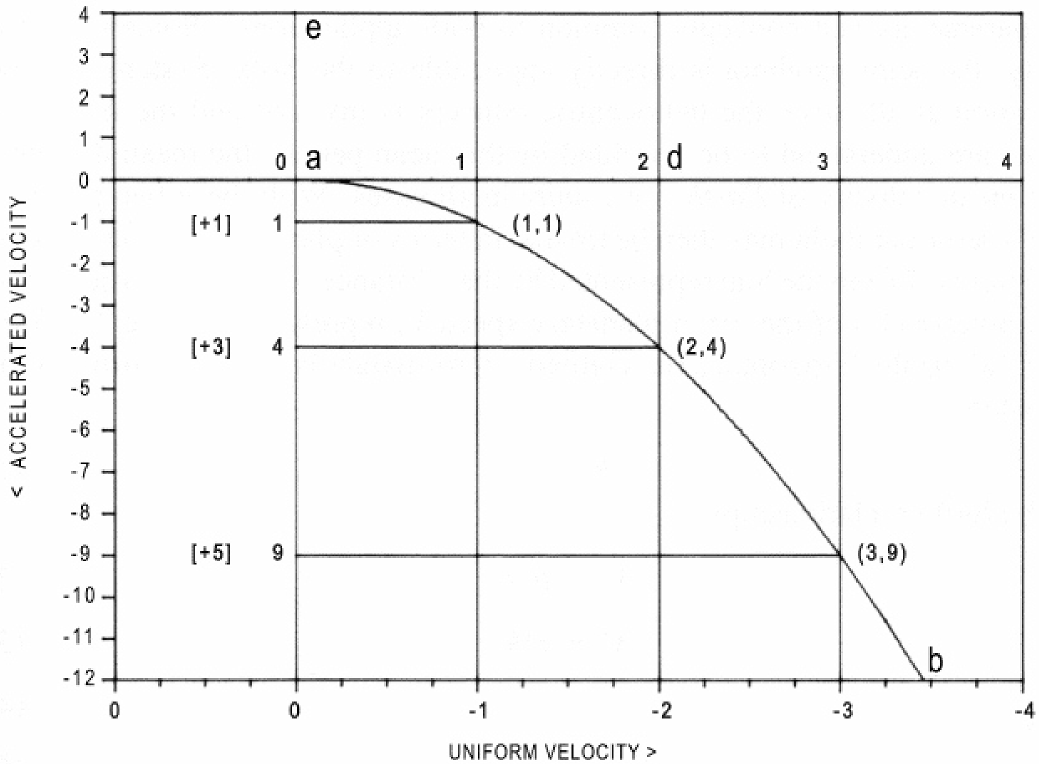

FIG. 1a—The Standard Parabola in normal context.

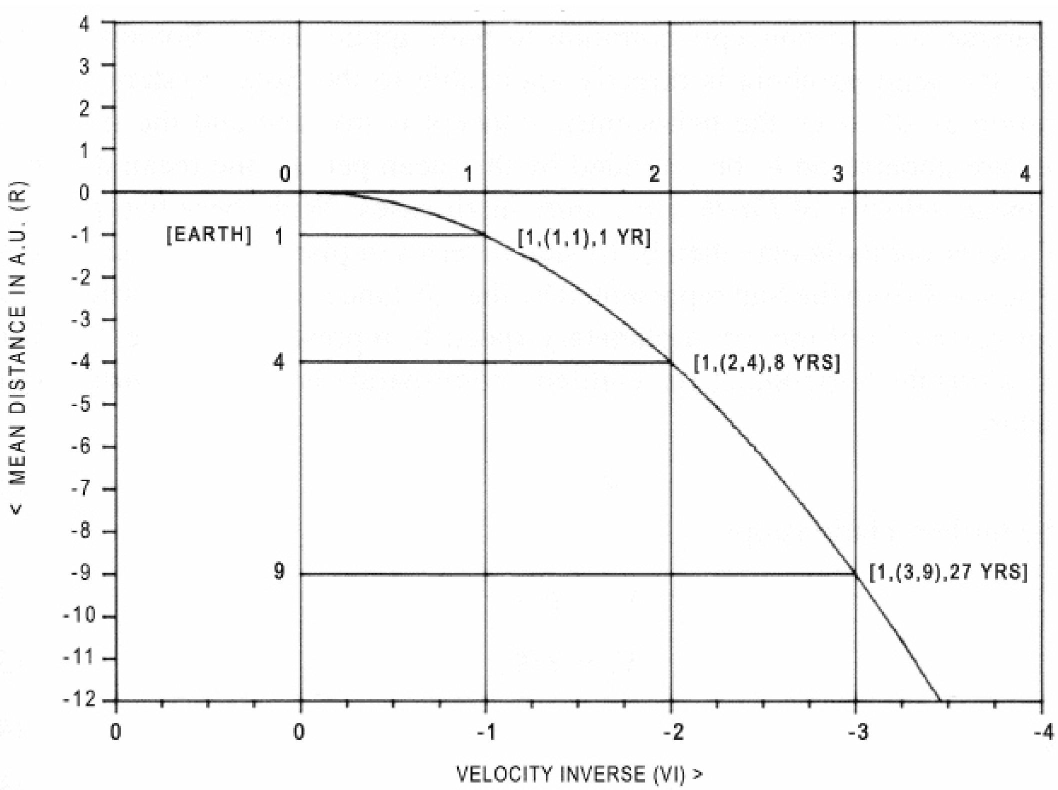

FIG. 1b—The Standard Parabola in astronomical context.

TABLE 1
MEAN PLANETARY DISTANCES, MEAN PERIODS AND MEAN VELOCITIES

Planet	Distance(R)	Period(T)	$V_r(R/T)$	$V_a(kR/T)$	Modern
Mercury	0.387107	0.24085	1.60725	47.88	47.89
Venus	0.723350	0.61521	1.17577	35.03	35.03
Earth	1	1	1	29.79	29.79
Mars	1.523733	1.88089	0.81011	24.13	24.13
Jupiter	5.201287	11.86223	0.43847	13.06	13.06
Saturn	9.538188	29.45770	0.32379	9.65	9.64
Uranus	19.182303	84.01390	0.22832	6.80	6.81
Neptune	30.057937	164.79300	0.18240	5.43	5.43
Pluto	39.440188	247.69000	0.15923	4.74	4.74

3. *The Parabola and Planetary Origins*. In spite of the limited treatment of the parabola in its astronomical context, it remains possible to hypothesize from material provided in *The New Sciences* and passages in his previous treatise, the *Dialogue Concerning the Two Chief World Systems*, that Galileo's analysis of the parabola and projectile trajectories could be expanded to a logical yet momentous conclusion.{p.212} Specifically,

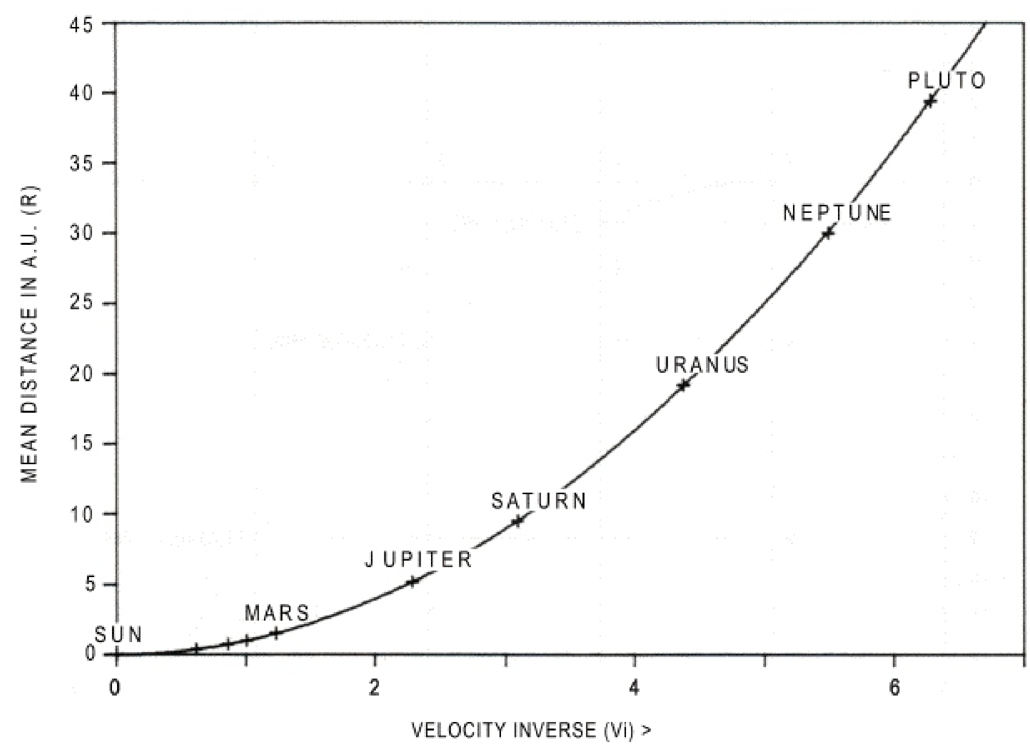

FIG. 2—The Parabola and the superior planets

his analysis of projectile paths could also conceivably be extended to encompass those special cases in which the projectiles "fall" into orbit about their parent bodies. In both terrestrial and astronomical applications the direction of "fall" may be understood to take place towards the *centre*, and indeed, in his earlier *Two Chief World Systems* Galileo had previously suggested in what is an undoubted heliocentric context:

> (that) among the decrees of the divine Architect was the thought of creating in the universe those globes which we behold continually revolving, and establishing *a centre of their rotations in which the sun was located immovably*. Next, suppose all the said globes to have been *created in the same place*, and their assigned tendencies of motion, *descending towards the centre* until they had acquired those degrees of velocity which originally seemed good to the Divine mind. *These velocities being acquired, we lastly suppose that the globes were set in rotation, each retaining in its orbit its predetermined velocity*. Now, *at what altitude and distance from the sun* would have been the place where the said globes were first created, and *could they have been created in the same place?*"(italics supplied)

while later, in *The New Sciences* he proposed that:

> ...God, after having created the heavenly bodies, assigned them the proper and uniform speeds with which they were forever to revolve... (and) *made them start from rest and move over definite distances under a natural and rectilinear acceleration such as governs the motion of terrestrial bodies... a body could not pass from rest to any given speed and maintain it uniformly except by passing through all the degrees of speed intermediate between the given speed and rest ... once these bodies had gained their proper and permanent speed, their rectilinear motion was converted into a circular one, the only motion capable of maintaining uniformity*, a motion in which the body revolves *without either receding from or approaching its desired goal.*" (italics supplied)

and finally asked with respect to the parabola:

> ... whether or not a *definite* 'sublimity' might be assigned to each planet, such that, if it were to *start from rest at this particular height and to fall with naturally accelerated motion along a straight line, and were later to change the speed thus acquired into uniform motion*, the size of the orbit and its period of revolution would be those actually observed." (italics supplied)

In the initial passage Galileo poses two questions: firstly whether the planets originated in one place, and secondly, whether the place in question can be identified. From a heliocentric viewpoint, because relative velocity decreases with distance from the Sun, one can understand how Galileo may have come to conceive that the planets originated with "zero" velocity beyond the region of Saturn (the outer limit of the Solar System in Galileo's era), but this does not address the question of origins itself. In the second passage in *The New Sciences*, however, these questions are accompanied by further amplifying details which also pertain to the fundamental parabola.[10]

Could Galileo have extended his treatment of terrestrial projectile paths to embrace satellite orbits and also have expanded the idea one step further to include the planets as satellites of the Sun? While acknowledging that there are dangers in attributing to Galileo modern or Newtonian concepts, it is necessary to recall that the initial discussion of the parabola concerned the path traced by a projectile with uniform horizontal velocity applied down the horizontal axis, and "naturally accelerated" velocity applied down the vertical axis. Visually, a projectile launched almost horizontally will obviously gain very little height before falling back to ground when the initial velocity is relatively low.[11] As the initial velocity increases, however, some height will be gained because of the curvature of the Earth, and although the projectile may still fall to ground, with sufficient velocity, a projectile will finally "fall" into orbit around Earth itself.[12] Thus in general, *by reversing matters, all objects in specific orbits may be treated in terms of a "percussive origins theory" with the parent body the initial source*. The hypothesis may therefore be applied to the planets and the Solar System with the *Sun* as the single percussive point of origin.[13]

Could Galileo have taken this final step? If he did, then undoubtedly criteria provided by Galileo in his historical aside becomes more significant than ever, i.e., if planetary origins are considered in terms of projectiles originating from the Sun, the planets would indeed "start with zero velocity" and "move through successive speeds" until their initial "rectilinear motion" changed into "circular motion" {p. 14} (or orbital motion) as they "fell" into their respective orbitals positions. And once established, the planets would then "revolve without either receding from or approaching" their common point of origin, or deviating from their "final" positions in the Solar System.

Although no causal mechanism is associated with this "percussive origins" (or "Small Bang") theory, the hypothesis might possibly assume that the Sun was essentially formed at this stage, and for whatever reason, the planetary material was ejected from the Sun in one enormous explosion.[14] In this sense the hypothesis is a variation of catastrophe theory, with the exception that the source of the catastrophe is internal rather than external. The latter, involving collisions or near misses with double or triple stars, etc., are not generally well supported today, but the percussive elements of the basic hypothesis may perhaps have some affinity with the massive explosion of the solar core (i.e., the "T Tauri winds") thought by some accretion theorists to be a possible explanation for the expulsion of unaccreted dust and gas from the Solar System.

4. Summary. Galileo's last work, the *Dialogues Concerning the New Sciences* followed his trial by the Inquisition for supporting the heliocentric concept in two previous publications. In addition to his subsequent imprisonment for this "crime", he was also required to recant his heretical views and refrain from any future discussion of the heliocentric hypothesis. In view of this restraint it is understandable that his treatise on projectile trajectories and their relationship to planetary velocities was both oblique and limited. Once alerted to his intentions, however, examination of *The New Sciences* indicates that Galileo did indeed manage to provide sufficient materials to permit his readers to apply the semi-parabola in the claimed terrestrial and astronomical contexts.[15]

The relative paucity of direct information notwithstanding, it has proved feasible to apply Galileo's semi-parabolas in the given astronomical context and understand the application in terms of the relationship between mean inverse speed Vi and mean planetary distance *R*, $R = V_i^2$ (see equation (1)). The distance *R* must be obtained initially from Kepler's Third Law,[16] which in turn may be combined with equation (1) to include the inverse of the velocity as in equation (5), and the velocity-based variants of the laws of planetary motion given by equations (2), (3) and (4).

For a given mean distance *R* or a given sidereal period *T* the absolute mean speed V_a and relative mean speed V_r may be obtained from equations (2) and (3), or from the following inverse square and inverse cube relations:

$$V_r = R^{-1/2} \qquad (6)$$

$$V_r = T^{-1/3} \qquad (7)$$

$$V_a = kR^{-1/2} \qquad (8)$$

$$V_a = kT^{-1/3} \qquad (9)$$

{p. 15}

The fundamental understanding and application of the semi-parabola in the astronomical context depends on the heliocentric concept, Kepler's Third Law for the mean distances, and relation (1). Although absolute confirmation may be lacking, it seems likely that Galileo - the originator of the material in its dual contexts - would have known, or would have been able to derive (in one form or another) all the velocity expansions of the laws of planetary motion given here.

The "percussive origins" theory of planetary motion also credited in this work to Galileo may perhaps be open to alternative interpretations, but the transition from projectile trajectories to satellite orbits is nevertheless a logical one. In view of his pioneering researches in the former area, and his discoveries in the other (the four Jovian moons which bear his name) such a progression would seem in keeping with both his heliocentric orientation and the general directions of his research.

Finally, three and a half centuries have passed since Galileo published the *Dialogues Concerning The New Sciences*. Apart from Mersenne's negative assessment of related concepts,[17] scant attention seems to have been paid to Galileo's oblique treatment of planetary velocities and planetary origins. Although his research may have been overshadowed by the works of Kepler and Newton, it seems that the obscure methodology forced on Galileo was if anything, only too successful. One cannot but help admiring Galileo's tenacity, however, for *The New Sciences* was written when he was in his seventies, with failing eyesight, and under the threat of most dire consequences should he ever attempt to discuss the heliocentric hypothesis again. Galileo may or may not have claimed at the conclusions of his trial that the Earth still moved, but it appears from the material in the *Dialogues Concerning The New Sciences* that he had the last word on the matter after all.

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APPENDIX

A1. The following dialogue between Salviati and Sagredo occurs in association with Galileo's "standard" parabola and an unexpected expansion that includes Plato, planetary motion, and the Solar System. (Fourth Day, [282-283] pp.259-260).

SALVIATI. But before we proceed further, since this discussion is to deal with the motion compounded of a uniform horizontal one and one accelerated vertically downwards - the path of a projectile, namely, a parabola - it is necessary that we define some common standard by which we may estimate the velocity, or momentum of both motions; and since from the innumerable uniform velocities one only, and not selected at random, is to be compounded with a velocity acquired by naturally accelerated motion, I can think of no simpler way of selecting and measuring this than to assume another of the same kind. For the sake of clearness, draw the vertical line *ac* to meet the horizontal line *bc*. *Ac* is the height and *bc* is the amplitude of the semi-parabola *ab*, which is the resultant of the two motions, one that of a body falling from rest at *a*, through the distance *ac*, with naturally accelerated motion, the other a uniform motion along the horizontal *ad*. The speed acquired at c by a fall through the dist ance *ac* is determined by the height *ac*; for the speed of a body falling from the same elevation is always one and the same; but along the horizontal one may give a body an infinite number of uniform speeds. However, in order that I may select one out of this multitude and separate it from the rest in a perfectly definite manner, I will extend the height *ca* upwards to *e* just as far as is necessary and will call this distance *ae* the "sublimity." Imagine a body to fall from rest at *e;* it is clear that we may make its terminal speed at a the same as that with which the same body travels along the horizontal line *ad*; this speed will be such that, in the time of descent along *ea*, it will describe a horizontal distance twice the length of *ea*. This preliminary remark seems necessary. The reader is reminded that above I have called the horizontal line *cb* the " amplitude " of the semi-parabola *ab*; the axis *ac* of this parabola, I have called its " altitude "; but the line *ea* the fall along which determines the horizontal speed I have called the " sublimity. " These matters having been explained, I proceed with the demonstration.

SAGREDO. *Allow me, please, to interrupt in order that I may point out the beautiful agreement between this thought of the Author and the views of Plato concerning the origin of the various uniform speeds with which the heavenly bodies revolve*. The latter chanced upon the idea that a body could not pass from rest to any given speed and maintain it uniformly except by passing through all the degrees of speed intermediate between the given speed and rest. Plato thought that God, after having created the heavenly bodies, assigned them the proper and uniform speeds with which they were forever to revolve; and that He made them start from rest and move over definite distances under a natural and rectilinear acceleration such as governs the motion of terrestrial bodies. He added that once these bodies had gained their proper and permanent speed, their rectilinear motion was converted into a circular one, the only motion capable its desired goal. ... *This conception is truly worthy of Plato; and it is all the more highly prized since its undying principles remained hidden until discovered by our Author who removed from them the mask and poetical dress and set forth the idea in correct historical perspective*. In view of the fact that *astronomical science furnishes us such complete information concerning the size of the planetary orbits, the distances of these bodies from their centers of revolution, and their velocities*, I cannot help thinking that our Author (to whom this idea of Plato was not unknown) had some curiosity to discover whether or not a definite "sublimity" might be assigned to each planet, such that, if it were to start from rest at this particular height and to fall with naturally accelerated motion along a straight line, and were later to change the speed thus acquired into uniform motion, the size of the orbit and its period of revolution would be those actually observed.

SALVIATI. *I think I remember his having told me that he once made the computation and found a satisfactory correspondence with the observation. But he did not wish to speak of it, lest in view of the odium which his many new discoveries had already brought upon him,* this might be adding fuel to the fire. *But if anyone desires such information he can obtain it for himself from the theory set forth in the present treatment.*[18] (emphases supplied)

Notes

1. Galileo's obscure and limited treatment of this subject may be explained by the fact that the *Dialogues Concerning Two New Sciences* was written after his trial for heresy for supporting the heliocentric concept in two previous treatises. Following his conviction by the Inquisition in 1633 he was forced to recant and expressly forbidden to discuss the heliocentric hypothesis again, or suffer the penalties of relapse.

2 Thus the ratio of the vertical to the horizontal axis at the point of calibration is 2:1. For this parabola a uniform velocity of TWO on the horizontal axis corresponds to distance of FOUR on the vertical axis. The same parabola also appears to have been used by Galileo to illustrate his observation that "... a moving body starting from rest and acquiring velocity at a rate proportional to time, will during equal intervals of time, traverse distances which are related to each other as the odd numbers beginning with unity, 1, 3, 5; or considering the total space traversed, that covered in double time will be quadruple that covered during unit time in triple time, the space is nine times as great as in unit time. And in general the spaces traversed are in the duplicate ratio of the times, i.e., in the ratio of the squares of the times." *The Two New Sciences*, Third Day [211-212]; also discussed in the Fourth Day [272-273].

3 Galileo follows Plato in his effective use of the Dialectic Method. The passages dealing with the parabola and the historical aside which follows are given here in the Appendix.

4 *The Dialogues Concerning Two New Sciences*, Fourth Day [283-284].

5 For example, Galileo discusses the sets: [1,2,4,8] and [1,3,9,27] with respect to squaring and cubing in the *Two Dialogues Concerning Two New Sciences* (First Day [83]). The same sets are also mentioned by Plato in the *Timaeus* (35b and 43d) and the first set [1,2,4,8] is discussed again in the *Epinomis* (991a-992a).

6 Matters are greatly simplified if mean circular motion is assumed, i.e., if the velocity of Earth is expressed in terms of the distance moved around the circumference divided by the mean period of revolution: 2Pi1/1 = 2Pi then the ratios of the mean velocities of the planets with respect to that of Earth will also reduce to ratios of mean distances divided by mean periods of revolution, i.e., 2PiR/T divided by 2Pi = R/T, and $V_a = KR/T$ etc.

7 Also the ancient relationship between a point, a line, an area, and a volume. See Galileo's discussion of the latter pair and the "sesquialteral ratio" between them in the *Two New Sciences*, First Day, (134-135).

8 The *Dialogue Concerning the Two Chief World Systems*, translated by Stillman Drake, 1967, p.29.

9 The *Dialogues Concerning The New Sciences*, Fourth Day, (282-283), translated by Henry Crew and Alphonso de Salvio, 1914, pp. 259-260. The "sublimity" may be understood to correspond to the distance between the vertex and the directrix for the parabola in both terrestrial and astronomical contexts.

10 I perhaps place too much significance on this point, but it does seem, in the last reference at least, that Galileo requires *a common, yet specific point of origin* with respect to each of the planets *and* the parabola. It is relevant to note here that the rotation of Earth is not directly involved in this application, although Galileo's views on this subject are of interest; for details see Stillman Drake's "Galileo and the Projection Argument," *Annals of Science*, 43, (1986), pp. 77-79.

11 Galileo discusses horizontal, near-horizontal projectile trajectories, and the parabola near the end of the *Dialogues Concerning The New Sciences in the Fourth Day*, (309-321).

12 But even though Galileo *could* have extended his work to this final conclusion, it should nevertheless still be acknowledged

that it is at odds with what is generally known concerning these aspects of Galileo's physics.

13 At least from a theoretical point of view or simple exercise, capture and accretion theories not excluded; to generalize, one might also include origins in other known satellite systems, even perhaps those of Jupiter and Saturn.

14 Or more than one single explosion.

15 Galileo seems to have supplied at least three alternative paths to reach this goal; once attained the rest follows almost as a matter of course.

16 The First and Second Laws of planetary motion are found in Johannes Kepler's *New Astronomy* published in 1609; the materia containing the Third (or the Harmonic) Law occurs in his Harmony of the Worlds published in 1618, some twenty years prior to the publication of Galileo's *Dialogues Concerning The New Sciences*. Applying Kepler's Third law in Galileo's astronomical application of the semi-parabola therefore causes no difficulties historically. Galileo's adherence to mean circular motion is also relevant in this same context.

17 *Harmonie Universalle*, second livre des mouvements, prop. 6.p.103, Paris, 1936. The uncritical acceptance of Mersenne's analysis appears to have unduly influenced subsequent commentators.

18 The *Dialogues Concerning The New Sciences*, Fourth Day, [282-283] translated by Henry Crew and Antonio de Salvio, 1914, pp. 259-260. the "sublimity" (the distance *ae*) may be understood to be the distance between the vertex and the directrix for the parabola in both terrestrial and astronomical contexts.

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